Branes at toric conical singularities*

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Abstract

After a brief survey of results related to the application of the AdS/CFT correspondence to $\mathcal{N} = 1$ supersymmetric models, I elaborate on certain geometric problems arising in this setup, more particularly on the construction of a Ricci-flat metric on the cone over a del Pezzo surface of rank one.

1. The AdS/CFT correspondence

The AdS/CFT conjecture states that certain conformal field theories (CFT) are in a precise way dual to string theory models describing string propagation in anti-de-Sitter space (AdS). The original example is the duality between the maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM) and the type IIB superstring in the space $AdS_5 \times S^5$ [1]. Consider Yang-Mills theory with gauge group $SU(N)$ and coupling constant g_{YM}. From the point of view of the $AdS_5 \times S^5$ geometry these two parameters – N and g_{YM} – are related to the radius of the sphere and the flux of the self-dual 5-form through it. [1]:

$$R^2 \sim \sqrt{g_{\mathrm{YM}}^2 N}, \qquad \int_{S^5} F_5 \sim N. \tag{1}$$

The correspondence between the two theories manifests itself, in particular, in the fact that the global symmetries of field theory correspond to the super-isometries of $AdS_5 \times S^5$. Indeed, the superconformal group of the $\mathcal{N} = 4$ theory is $PSU(2,2|4)$. Its maximal bosonic subgroup is $SU(2,2) \times$

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[1] We note that, since the form F_5 is self-dual, it has nonzero components along the AdS space as well.

$SU(4)$. Here the first factor is isomorphic to $SO(2,4)$ – the conformal group of four-dimensional Minkowski space, and the second one represents the R-symmetry group of the theory. On the other hand, $SO(2,4)$ and $SO(6)$ are isometry groups of the anti-de-Sitter space and the five-sphere, respectively.

The simplest modification of this basic model is obtained if one factorizes the sphere by a discrete subgroup of the isometry group $SO(6)$:

$$AdS_5 \times S^5 \longrightarrow AdS_5 \times S^5/\Gamma, \qquad \Gamma \subset SO(6). \tag{2}$$

The dual gauge theory is now different: the gauge group consists of several simple factors, and the matter multiplets are in bifundamental representations. The number of supersymmetries is also reduced, and it depends on the structure of the group Γ.

2. Sasakian manifolds and $\mathcal{N} = 1$ supersymmetry

All manifolds of the type $AdS_5 \times S^5/\Gamma$ are particular cases of a rather wide class of solutions of IIB supergravity of the form $AdS_5 \times X^5$, in which the dilaton ϕ is constant and the only nonzero form is the 5-form:

$$\phi = \text{const.}, \qquad F_5 \sim N\left((\text{vol})_{AdS_5} + (\text{vol})_{X^5}\right). \tag{3}$$

The Bogomolnyi-Prasad-Sommerfeld condition for this configuration, i.e. the requirement of preservation of at least one supersymmetry, reduces to the vanishing of the gravitino variation:

$$\delta\psi_\mu = (\nabla_\mu + F_5\gamma_\mu)\epsilon = 0. \tag{4}$$

The spinor ϵ, satisfying (4), is called a Killing spinor. The existence of a Killing spinor substantially constrains the geometry of the space X^5. In particular, one can show, similarly to the way it is done in [2], that the solubility of the equation (4) is equivalent to the requirement that the metric cone over X^5 should be Kähler and Ricci-flat. In other words, let $(\widetilde{ds^2})_{X^5}$ be the metric on X^5. The metric on the cone is, by definition, $ds^2 = dr^2 + r^2(\widetilde{ds^2})_{X^5}$. The requirement of supersymmetry is that the metric ds^2 is Kähler and Ricci-flat (the latter following as well directly from the fact, that due to the supergravity equations of motion and the 5-form (3) $(\widetilde{ds^2})_{X^5}$ is an Einstein metric of positive curvature). In this case the metric on the base of the cone $(\widetilde{ds^2})_{X^5}$ is called Sasaki-Einstein.

Locally all Sasaki-Einstein metrics look rather simple: they can be written in the form

$$(ds^2)_{X^5} = (d\varphi - A)^2 + \sum_{i,j=1}^{2} g_{i\bar{j}}\, dz_i\, d\bar{z}_j. \tag{5}$$

Here $g_{i\bar{j}}$ is a Kähler-Einstein metric on some complex surface (the meaning of this surface is explained below, see section 2.1.1.) and A is the Kähler current. Validity of such representation is clear for the sphere S^5, if one recalls that it is the total space of a Hopf fiber bundle with base $\mathbb{CP}^2$. An important difference of the general case is that $g_{i\bar{j}}$ does not necessarily have to be smooth (although the metric on X^5 is smooth).

On a Sasaki-Einstein manifold with the metric (5) there is a canonical vector field $\partial/\partial\varphi$ with fixed norm – it is the so-called Reeb vector. From the point of view of the dual gauge theory, this $U(1)$ isometry of the space X^5 is dual to the global $U(1)$ symmetry of the superconformal field theory – the R-symmetry, which acts on the supercharges [3]:

$$Q \to e^{i\alpha} Q, \qquad \bar{Q} \to e^{-i\alpha} \bar{Q}. \tag{6}$$

In contrast to a generic theory with $\mathcal{N} = 1$ supersymmetry, a superconformal theory always possesses a $U(1)$ R-symmetry, since the generator of R-symmetry enters explicitly in the superalgebra $SU(2,2\,|\,1)$ (hence the R-transformations are its inner automorphisms).

2.1. Calabi-Yau manifolds and singularities

The construction of the cone over a Sasaki-Einstein manifold X^5, described above, allows one to take a new look at the configuration of the type $AdS_5 \times X^5$. Here one can recall the interpretation of the $\mathcal{N} = 4$ SYM theory as an effective field theory describing the oscillations of a stack of parallel D3-branes embedded in flat space $\mathbb{R}^{1,9}$. Clearly, in this case the transverse space to the branes is $\mathbb{R}^6$. The branes are massive objects, and therefore they change the geometry of the space, in which they are embedded. Assuming that the branes are located at the origin of the space $\mathbb{R}^6$, one can introduce a radial coordinate r and look for supergravity solutions of the type

$$ds^2 = h^{-1/2}(r) \sum_{i=1}^{4} dx_i^2 + h^{1/2}(r)(dr^2 + r^2 (d\Omega)_{S^5}). \tag{7}$$

Since the branes are charged with respect to the 5-form F_5, there is an extra condition $\int_{S^5} F_5 \sim N$. The solution has the form [4] (l is a linear scale)

$$h(r) = 1 + \frac{l^4\, N}{r^4}. \tag{8}$$

In the limit $r \to 0$ we obtain the metric of $AdS_5 \times S^5$. It is clear then that the sphere S^5 emerges as the locus of points, equidistant from $r = 0$ (in the natural flat metric) in the six-dimensional space, in which the branes are embedded. Now we can consider the case when the 'internal' six-dimensional space is not a flat one, but rather a general (compact) Calabi-Yau space Y^6. Placing the branes at a nonsingular point of Y^6, we will again have $\mathcal{N} = 4$ SYM as a low-energy limit. The situation changes, however, if we place the branes at a singular point of the Calabi-Yau space. In this case the effective field theory on the branes depends on the local geometry of Y^6 in the vicinity of the singularity. The neighborhood of the singularity can be described by a noncompact Ricci-flat metric of the conical type, i.e. the metric which has the form $(ds^2)_{sing} = dr^2+r^2(\widetilde{ds^2})_{X^5}$. In this case the points equidistant from the singularity form not an S^5 but rather the Sasaki-Einstein manifold X^5. The ansatz (7) is still valid and leads in the limit $r \to 0$ to the configuration $AdS_5 \times X^5$ [5].

2.1.1. The cones

It follows from the previous section that the spaces X^5 are tightly connected with the singularities of the Calabi-Yau manifolds Y^6 of complex dimension 3. Here we will focus on those singularities which are singularities of complex cones over complex surfaces. The surfaces will be denoted by $\mathcal{M}$ hereafter. However, it is simplest to explain the notion with the example of a cone over a complex curve, namely when $\mathcal{M} = \mathbb{CP}^1$. Indeed, consider a conic in $\mathbb{CP}^2$:

$$X_0^2 + X_1^2 + X_2^2 = 0 \tag{9}$$

This algebraic variety is isomorphic to $\mathbb{CP}^1$, and it is nonsingular as a hypersurface in $\mathbb{CP}^2$. In order to pass to the cone, we should 'forget' that X_0, X_1, X_2 are projective coordinates, that is to say we will treat them as ordinary affine variables. In other words, we will consider the equation (9) as defined in $\mathbb{C}^3$. Thus one obtains a singularity at the origin $X_0 = X_1 = X_2 = 0$. We will call this singular variety the complex cone over $\mathbb{CP}^1$. This definition can be extended in a straightforward way to complex cones over complex surfaces (one just needs to consider a higher-dimensional ambient space $\mathbb{CP}^N$ and certain algebraic equations therein).

The only remaining question is in which case the affine manifolds of the sort (9) are Ricci-flat (or, more precisely, in which case they remain Ricci-flat after the singularity at the origin has been resolved). It turns out that this requirement is met if and only if the complex surface $\mathcal{M}$ has an ample anticanonical bundle. (In the example (9) one is in fact dealing with a complex curve $\mathbb{CP}^1$, which has an ample anticanonical bundle as well.) In the language of differential geometry it means that $\mathcal{M}$ is a positively curved surface, or, more exactly, that the integral of the Ricci form over any homologically nontrivial two-cycle C is positive:

$$\int_C \frac{i}{2\pi} R_{m\bar{n}}\, dz_m \wedge d\bar{z}_n > 0. \tag{10}$$

It is known from algebraic geometry that the only compact nonsingular surfaces of positive curvature are $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$, as well as the blow-ups of $\mathbb{CP}^2$ in no more than eight points – these are the so-called del Pezzo surfaces (two-dimensional Fano varieties).

Pick one of these surfaces $\mathcal{M}$. How can one build a corresponding Sasaki-Einstein metric on X^5? When $\mathcal{M}$ admits a Kähler-Einstein metric (with Kähler potential K), the answer is given by formula (5). It can be rewritten in a slightly different way, if one recalls the correspondence between the Kähler-Einstein metrics on $\mathcal{M}$ and Ricci-flat metrics on Y^6. One can search for a Kähler potential $\mathcal{K}$, which defines the (Ricci-flat) metric on the complex cone over $\mathcal{M}$, in the following form:

$$\mathcal{K} = \mathcal{K}(|u|^2 e^K). \tag{11}$$

This ansatz is known as Calabi's ansatz [6].

3. Cone over the del Pezzo surface

The del Pezzo surface of rank one, $\mathbf{dP}_1$, is the blow-up of $\mathbb{CP}^2$ at one point. There is a concrete algebraic model for it. The surface can be embedded into $\mathbb{CP}^8$, and the embedding is given by those sections of $\mathcal{O}_{\mathbb{CP}^2}(3)$ which vanish at a given point on $\mathbb{CP}^2$, for example at $(z_1 : z_2 : z_3) = (0 : 0 : 1)$. The embedding is called *anticanonical*, since the standard tautological sheaf $\mathcal{O}_{\mathbb{CP}^8}(1)$ over the ambient space $\mathbb{CP}^8$, when restricted to the surface, coincides with its anticanonical sheaf. Once the embedding is specified, the *affine cone* may be constructed simply by passing from projective space $\mathbb{CP}^8$ to the affine space $\mathbb{C}^9$.

In this construction we have chosen a reference point $(0 : 0 : 1) \in \mathbb{CP}^2$, which was subsequently blown-up. This reduces the automorphism group $PGL(3, \mathbb{C})$ of $\mathbb{CP}^2$ to the automorphism group of $\mathbf{dP}_1$:

$$Aut(\mathbf{dP}_1) = P \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{pmatrix}, \tag{12}$$

and the cone $Y := \text{Cone}(\mathbf{dP}_1)$ has as its automorphism group the maximal parabolic subgroup of $GL(3, \mathbb{C})$ defined by matrices of the form (12) (forgetting the projectivization).

We will be looking for a Kähler metric on Y with the isometry group being the maximal compact subgroup of $Aut(Y)$:

$$\text{Isom}(Y) = U(2) \times U(1) \tag{13}$$

In more practical terms, we will introduce three complex coordinates z_1, z_2, u on Y and, due to the $U(2) \times U(1)$ isometry, we will assume that the Kähler potential depends on the two combinations of them:

$$K = K(|z_1|^2 + |z_2|^2, |u|^2) \tag{14}$$

The corresponding Kähler form is $\Omega = i\partial\bar{\partial}K$ and the metric is $g_{i\bar{j}} = \partial_i \bar{\partial}_j K$. The Ricci tensor is related to the metric of a Kähler manifold as $R_{i\bar{j}} = -\partial_i \bar{\partial}_j \log \det g$. The Ricci-flatness (Calabi-Yau) condition $R_{i\bar{j}} = 0$ takes the form of a Monge-Ampere equation for the function $G(\mu, \nu)$, which is the

Legendre transform of K with respect to the variables $t = \log\left(|z_1|^2 + |z_2|^2\right)$ and $s = \log\left(|u|^2\right)$:

$$G = \mu\, t + \nu\, s - K \tag{15}$$

The usefulness of the new variables (μ, ν) to a large extent relies on the fact that they have a transparent geometric meaning – these are the moment maps for the following two $U(1)$ actions on Y:

$$U(1)_\mu : \quad (\, z_1 \to e^{i\alpha}\, z_1, \quad z_2 \to e^{i\alpha}\, z_2\,) \qquad\qquad U(1)_\nu : \quad u \to e^{i\beta}\, u \tag{16}$$

The Ricci-flatness equation has the following form:

$$e^{\frac{\partial G}{\partial \mu} + \frac{\partial G}{\partial \nu}} \left(\frac{\partial^2 G}{\partial \mu^2} \frac{\partial^2 G}{\partial \nu^2} - \left(\frac{\partial^2 G}{\partial \mu \partial \nu} \right)^2 \right) = \tilde{a}\, \mu \tag{17}$$

Denoting (μ, ν) by (μ_1, μ_2), we can recover the metric from the dual potential G [7] using the formula

$$ds^2 = \mu\, g_{\mathbb{CP}^1} + \sum_{i,j=1}^{2} \frac{\partial^2 G}{\partial \mu_i \partial \mu_j}\, d\mu_i\, d\mu_j + \sum_{i,j=1}^{2} \left(\frac{\partial^2 G}{\partial \mu^2} \right)^{-1}_{ij} (d\phi_i - A_i)\, (d\phi_j - A_j)\,, \tag{18}$$

where $g_{\mathbb{CP}^1}$ is the standard round metric on $\mathbb{CP}^1$, $A_2 = 0$ and A_1 is the 'Kähler current' of $\mathbb{CP}^1$, i.e. a connection, whose curvature is the Fubini-Study form of $\mathbb{CP}^1$: $dA_1 = \omega_{\mathbb{CP}^1}$.

Since (μ, ν) are moment maps for the $U(1)^2$ action, the domain on which the potential $G(\mu, \nu)$ is defined is the moment polygon for this $U(1)^2$ action. In this case it is an unbounded domain with two vertices, hence we may call it a 'biangle'.

From the perspective of the equation (17), it is the singularities of the function G that determine the polygon. As we approach an arbitrary edge L_i of the polygon, i.e. when $L_i \to 0$, we impose the asymptotic condition

$$G = L_i\, (\log L_i - 1) + \ldots \quad \text{as} \quad L_i \to 0, \tag{19}$$

where the ellipsis indicates terms regular at $L_i \to 0$. Despite being subleading, they are important for the equation (17) to be consistent even in the limit $L_i \to 0$.

Notice that, in addition to the $U(1)^2$ action (16), there is yet another $U(1)$ action given by $(z_1 \to e^{i\alpha}\, z_1,\ z_2 \to e^{-i\alpha}\, z_2)$. Therefore the fiber over a generic point of the biangle is $\mathbb{CP}^1 \times \mathbb{T}^2$. The angles of the moment polygon are detemined by the normal bundles to the two $\mathbb{CP}^1$'s 'located' in the corners (see [8] for a detailed discussion).

3.1. An expansion away from the vertex of the cone

To start the analysis of the equation (17) first of all we shift the origin along the μ-axis by a constant μ_0 in such a way that the new origin is located at the intersection point of the two outer lines of the moment 'biangle'.

We aim at building an expansion of the metric at 'infinity', i.e. far from the 'vertex'. For this purpose, instead of the $\{\mu, \nu\}$ variables, we will use a 'radial' variable ν and an 'angular' variable ξ:

$$\{\mu, \nu\} \to \left\{\nu,\ \xi = \frac{\mu - \mu_0}{\nu}\right\} \tag{20}$$

Then the equation (17) above may be rewritten as follows:

$$e^{\frac{\partial G}{\partial \nu} - \frac{\xi - 1}{\nu}\frac{\partial G}{\partial \xi}} \left[\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 G}{\partial \nu^2} - \left(\frac{\partial^2 G}{\partial \xi \partial \nu} - \frac{1}{\nu}\frac{\partial G}{\partial \xi}\right)^2\right] = a\,\nu^3\left(\xi + \frac{\mu_0}{\nu}\right) \tag{21}$$

We propose the following expansion for the potential G at $\nu \to \infty$ (b is a constant):

$$G = 3\nu(\log\nu - 1) + \nu\,P_0(\xi) + b\log\nu + \sum_{k=0}^{\infty} \nu^{-k}\,P_{k+1}(\xi) \tag{22}$$

Substituting this expansion in the equation, we obtain a 'master' equation, which can then be expanded in powers of $\frac{1}{\nu}$ and solved iteratively for the functions $P_k(\xi)$.

3.1.1. Leading order

The first equation is obtained from (21) in the limit $\nu \to \infty$:

$$P_0'' = \frac{a}{3}\,\xi\,e^{(\xi-1)P_0' - P_0} \tag{23}$$

and has the solution

$$P_0(\xi) = \log\left(-\frac{a}{9}\right) - \sum_{i=0}^{2}\frac{\xi - \xi_i}{\xi_i - 1}\ \log(\xi - \xi_i), \tag{24}$$

where ξ_i are the roots of the polynomial

$$Q(\xi) = \xi^3 - \frac{3}{2}\xi^2 + d, \tag{25}$$

and d is a constant of integration, which plays a crucial geometric role.

The function $P_0(\xi)$ determines the metric at infinity by means of the formulas (22) and (18). One can check that in the original (μ, ν) variables the 'radial' part of the metric is determined by

$$G_0 = \sum_{i=0}^{2} \frac{\mu - \xi_i\, \nu}{1 - \xi_i}\, \left(\log\left(\mu - \xi_i\, \nu\right) - 1\right) \tag{26}$$

It turns out that there is a topological restriction on the roots ξ_i, which is related to the fact that the two $\mathbb{CP}^1$'s embedded in the corners of the moment 'biangle' should have correct normal bundles (a full discussion of this can be found in [8]). As a result the value of d in (25) is uniquely fixed to

$$d = \frac{16 + \sqrt{13}}{64}\,. \tag{27}$$

Quite generally, it may be shown that the function G satisfying eq. (21) has the following structure (up to a certain 'regularity requirement' at the edges of the moment biangle, see [8]):

$$G = \frac{\tilde{\mu} - \xi_1\, \nu}{1 - \xi_1} \left(\log\left(\frac{\tilde{\mu} - \xi_1\, \nu}{1 - \xi_1}\right) - 1\right) + \frac{\tilde{\mu} - \xi_2\, \nu}{1 - \xi_2} \left(\log\left(\frac{\tilde{\mu} - \xi_2\, \nu}{1 - \xi_2}\right) - 1\right) + \\ + \left(\frac{\tilde{\mu} - \xi_0\, \nu}{1 - \xi_0} + b\right) \left(\log\left(\frac{\tilde{\mu} - \xi_0\, \nu}{1 - \xi_0} + b\right) - 1\right) + b \sum_{k=2}^{\infty} \left(\frac{b}{\nu}\right)^k \tilde{P}_{k-2}(\xi),$$

where $\tilde{P}_k(\xi)$ is a polynomial of degree k. Here the variable μ has been shifted in such a way that the new origin is located at $\tilde{\mu} = \nu = 0$.

3.1.2. Singular points of the Heun equation and eigenfunctions

We proceed to describe in more detail the equations that arise in higher orders of perturbation theory. In the k-th order we arrive at the following equation:

$$D_k \tilde{P}_k := \frac{d}{d\xi} \left(Q(\xi) \frac{d\tilde{P}_k}{d\xi}\right) - \left((k+1)^2 - 1\right)\, \xi \tilde{P}_k = \text{r.h.s.}, \tag{28}$$

where $Q(\xi)$ has been defined in (25) and the right hand side depends on the previous orders of perturbation theory, i.e. on $\tilde{P}_{k-1}, \ldots, \tilde{P}_0$ and their derivatives. One can show that the inhomogeneous equation (28) has a polynomial solution of degree k. The general solution, however, is produced by adding to this particular solution a general solution of the homogenized equation $D_k \Pi_k = 0$. The roots $\xi_i, i = 0, 1, 2$ of the polynomial $Q(\xi) = \prod_{i=0}^{2} (\xi - \xi_i)$ are singular points of this equation. Moreover, by making the change of variables $\xi \to \frac{1}{\xi}$, one easily sees that ∞ is a singular point as well. Hence $D_k \Pi_k = 0$ is a Fuchsian equation with 4 singular points – a particular case of the so-called Heun equation, in which all exponents are zero.

The question we wish to pose is whether the homogenized equation $D_k \Pi_k = 0$ has a nontrivial solution regular at *two* of the singular points, say ξ_1, ξ_2. This is necessary in order to comply with the regularity requirement mentioned above. We claim that the answer is positive only for $k = 0, 1$:

$$\Pi_0 = \alpha \tag{29}$$

$$\Pi_1 = \beta(\xi - 1), \tag{30}$$

$$\text{where} \quad \alpha, \beta = \text{const.}$$

Quite interestingly, the nontrivial solutions are independent of the constant d, which suggests that they are also relevant for the deformations of other Ricci-flat cones asymptotic to (real) cones over Sasaki-Einstein manifolds.

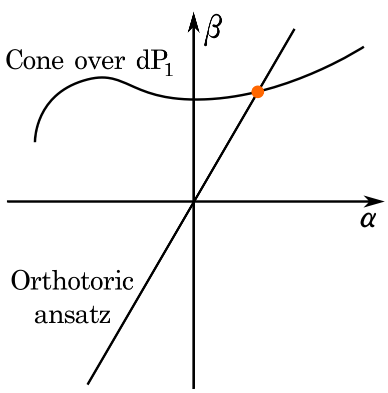

Figure 1: The red spot represents the known (orthotoric) metric on the cone over $\mathbf{dP}_1$.

In general the fact that some parameters are absent in the metric at infinity and then appear in different orders of the expansion around this metric is compatible with the known cases. One prominent example is the resolved conifold, which is asymptotic to the real cone over the Einstein-Sasaki manifold $T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$ at infinity and exhibits two resolution parameters in an expansion around infinity [9].

As we have just discussed, there exists a Ricci-flat metric with $U(2)\times U(1)$ isometry on the complex cone over $\mathbf{dP}_1$ with *at most* two parameters, which we termed α and β. There exists a closed expression for G, and hence for the metric, in a particular case when the parameters α and β are related in a certain way — this is the metric obtained in [10], as well as in [11] by means of the so-called 'orthotoric' ansatz developed in [12].

4. Conclusion and outlook

We have explained, how remarkable geometric objects appear in the analysis of $\mathcal{N} = 1$ superconformal theories via the AdS/CFT correspondence. As a particular interesting example, we have analyzed the parameter space of Ricci-flat metrics on the complex cone over a del Pezzo surface of rank one (sometimes also called the Hirzebruch surface F_1). In particular, using an expansion at infinity, we have found one potential new parameter β (see (30)). In general we conjecture that there is a particular relation between β and α that preserves the correct topology, i.e. $\beta = \beta(\alpha)$ (see Fig. 1).

In this case the remaining parameter is related to the size of the blown-up $\mathbb{CP}^1$ in the base of the cone, i.e. in the del Pezzo surface.

It would be very interesting to obtain an exact formula (like the one of [12]-[11]), for the solution with two generic values of the parameters α, β, and this would certainly shed light on these questions.

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