Fundamental issues in extended geometry

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Abstract

This talk gives the basics of extended geometry, with a focus on the gauge transformations, the generalised diffeomorphisms. Some global issues are discussed. I also go into some detail about how the formalism can be made to allow for a geometric derivation of discrete duality transformations and of monodromies for non-geometric field configurations.

Duality symmetries in string theory/M-theory mix gravitational and non-gravitational fields. Manifestation of such symmetries calls for a generalisation of the concept of geometry.

It has been proposed that the compactifying space (torus) is enlarged to accommodate momenta (representing momenta and brane charges) in modules of a duality group. This leads to *doubled geometry* [1]-[23]. in the context of T-duality, and *exceptional geometry* [24]-[39] in the context of U-duality.

In the present talk, I will

- Describe the basics of extended geometry: fields, gauge transformations, &c.
- Discuss some global issues concerning generalised manifolds.
- Make precise how duality transformations become "geometric", and what remains for a full description.
- Point out some questions and directions.

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Consider a compactification from 11 to $11-n$ dimensions on T^n. As is well known, all fields and charges fall into modules of $E_{n(n)}$:

n	$E_{n(n)}$	R_1
3	$SL(3)\times SL(2)$	$(\mathbf{3},\mathbf{2})$
4	$SL(5)$	$\mathbf{10}$
5	$Spin(5,5)$	$\mathbf{16}$
6	$E_{6(6)}$	$\mathbf{27}$
7	$E_{7(7)}$	$\mathbf{56}$
8	$E_{8(8)}$	$\mathbf{248}$

I will focus on diffeomorphisms, and how they generalise. The ordinary diffeomorphisms go together with gauge transformations for the 3-form and (dual) 6-form fields (and for high enough n also gauge transformations for dual gravity) in an $E_{n(n)}$ module R_1. This is the "coordinate module". The derivative transforms in $\bar{R}_1$.

The situation for T-duality is simpler. Compactification from 10 to $10-d$ dimensions give the (continuous) T-duality group $O(d,d)$. The momenta are complemented with string windings to form the $2d$-dimensional module.

Note that the duality group is not to be seen as a global symmetry. Instead, discrete duality transformations in $O(d,d;\mathbb{Z})$ or $E_{n(n)}(\mathbb{Z})$ should arise as symmetries in certain backgrounds, just as the mapping class group $SL(n;\mathbb{Z})$ arises as discrete isometries of a torus. The rôle of the continuous versions of the duality groups should be analogous to that of $GL(n)$ in ordinary geometry (gravity).

One has to decide how tensors transform. The generic recipe is to mimic the Lie derivative for ordinary diffeomorphisms:

$$L_U V^m = U^n\partial_n V^m - \partial_n U^m V^n$$

where the first term is a transport term and the second a **gl** transformation. In the case of U-duality, the role of GL is assumed by $E_{n(n)}\times\mathbb{R}$, and

$$\begin{aligned}\mathscr{L}_U V^M &= L_U V^M + Y^{MN}{}_{PQ}\partial_N U^P V^Q\\ &= U^N\partial_N V^M + Z^{MN}{}_{PQ}\partial_N U^P V^Q\end{aligned}$$

where $Z^{MN}{}_{PQ} = -\alpha_n P^M_{\text{adj}}{}_Q,{}^N{}_P + \beta_n\delta^M_Q\delta^N_P = Y^{MN}{}_{PQ} - \delta^M_P\delta^N_Q$. projects on the adjoint of $E_{n(n)}\times\mathbb{R}$. Y is an invariant tensor, the form of which we do not give here (see ref. [34]).

The transformations form an algebra for $n \leq 7$:

$$[\mathscr{L}_U, \mathscr{L}_V]W^M = \mathscr{L}_{[\![U,V]\!]}W^M$$

where the "Courant bracket" is $[\![U,V]\!]^M = \frac{1}{2}(\mathscr{L}_U V^M - \mathscr{L}_V U^M)$, provided that the derivatives fulfill a "*section condition*".

This section condition ensures that fields locally depend only on an n-dimensional subspace of the coordinates, on which a $GL(n)$ subgroup acts. It reads $Y^{MN}{}_{PQ}\partial_M \ldots \partial_N = 0$, or

$$(\partial \otimes \partial)|_{\bar{R}_2} = 0$$

n	R_1	R_2
3	$(\mathbf{3}, \mathbf{2})$	$(\bar{\mathbf{3}}, \mathbf{1})$
4	$\mathbf{10}$	$\bar{\mathbf{5}}$
5	$\mathbf{16}$	$\mathbf{10}$
6	$\mathbf{27}$	$\bar{\mathbf{27}}$
7	$\mathbf{56}$	$\mathbf{133}$
8	$\mathbf{248}$	$\mathbf{1} \oplus \mathbf{3875}$

The interpretation of the section condition is that the momenta locally are chosen so that they may span a linear subspace of cotangent space with maximal dimension, such that any pair of covectors p, p' in the subspace fulfill $(p \otimes p')|_{\bar{R}_2} = 0$.

The corresponding statement in T-duality is $\eta^{MN}\partial_M \otimes \partial_N = 0$, where η is the $O(d,d)$-invariant metric. The maximal linear subspace is a d-dimensional isotropic (light-like) subspace, and it is determined by a pure spinor Λ. Once a Λ is chosen, the section condition can be written $\Gamma^M \Lambda \partial_M = 0$. An analogous linear construction can be performed in the exceptional setting [34].

The generalised diffeomorphisms do not satisfy a Jacobi identity. On general grounds, it can be shown that the "Jacobiator" is proportional to $(\!([\![U,V]\!], W)\!) + \text{cycl}$, where $(\!(U,V)\!) = \frac{1}{2}(\mathscr{L}_U V + \mathscr{L}_V U)$.

It is important to show that the Jacobiator in some sense is trivial. It turns out that $\mathscr{L}_{(\!(U,V)\!)}W = 0$ (for $n \leq 7$), and the interpretation is that it is a gauge transformation with a parameter representing reducibility.

In doubled geometry, this reducibility is just the scalar reducibility of a gauge transformation: $\delta B_2 = d\lambda_1$, with the reducibility $\delta\lambda_1 = d\lambda'_0$.

In exceptional geometry, the reducibility turns out to be more complicated, leading to an infinite (but well defined) reducibility, containing the modules of tensor hierarchies, and providing a natural generalisation of forms (having connection-free covariant derivatives).

I will skip the detailed description of the generalised gravity. It effectively provides the local dynamics of gravity and 3-form, which are encoded by a vielbein $E_M{}^A$ in the coset $(E_{n(n)} \times \mathbb{R})/K(E_{n(n)})$.

n	$E_{n(n)}$	$K(E_{n(n)})$
3	$SL(3) \times SL(2)$	$SO(3) \times SO(2)$
4	$SL(5)$	$SO(5)$
5	$Spin(5,5)$	$(Spin(5) \times Spin(5))/\mathbb{Z}_2$
6	$E_{6(6)}$	$USp(8)/\mathbb{Z}_2$
7	$E_{7(7)}$	$SU(8)/\mathbb{Z}_2$

The T-duality case is described by a generalised metric or vielbein in $O(d,d)/(O(d) \times O(d))$, parametrised by the ordinary metric and B-field.

With some differences from ordinary geometry, one can go through the construction of connection, torsion, metric compatibility &c., and arrive at generalised Einstein's equations encoding the equations of motion for all fields.

One may introduce global or local supersymmetry, although the concept of superfields and supergeometry is quite unexplored.

Just like a manifold may be described by an atlas of coordinate charts with transition functions on the overlaps, we want to patch a generalised manifold by overlaps, that must be *finite generalised diffeomorphisms.*

In ordinary geometry, the transition functions are matrices $M_M{}^N = \frac{\partial X^N}{\partial X'^M}$, and covectors obey

$$A'_M(X') = M_M{}^N A_N(X)\,.$$

Now we need M to be replaced by a group element F in $E_{n(n)} \times \mathbb{R}$ or $O(d,d)$:

$$A'_M(X') = F_M{}^N(M) A_N(X)\,.$$

The matrix F is known explicitly for $O(d,d)$ [16][18].

$$F(M) = \frac{1}{2}\left(M(M^{-1})^t + (M^{-1})^t M\right)$$

Only partial results exist for exceptional groups [40].

The $O(d,d)$ result can be obtained from exponentiation of the generalised Lie derivative. The naïve composition rule does not hold, $F(M)F(N) \neq F(MN)$, *i.e.*, the map $F : GL(2d) \to O(d,d)$ is not a group homomorphism. Instead, a "twisted" version holds [18],

$$F(M)F(N) = F(MN)e^{\Delta}\,,$$

where e^{Δ} is a generalised diffeomorphism that leaves the coordinated unchanged. The existence of such transformations is due to the section condition, and has no counterpart in ordinary geometry. Such a "non-translating" generalised coordinate transformation only transforms the B-field, and not the metric (given an explicit solution to the section condition).

The situation can be summarised as follows: For any choice of $M = \frac{\partial X}{\partial X'}$ there is an equivalence class of generalised diffeomorphisms, all given by $F(M,\Delta) = F(M)e^{\Delta}$ for some Δ, with $F(M)$ as a canonical representative. The map from $GL(2n)$ to the equivalence class *is* a homomorphism.

This leads to a gerbe structure. Defining

$$H(M,N) = F(M)F(M^{-1}N)F(N^{-1})\,,$$

the product

$$\Lambda(M,N,P) = H(M,N)H(N,P)H(P,M)$$

defines the non-trivial triple overlap (cocycle).

It is not surprising that a gerbe structure arises, given that tensor gauge transformations are contained in the formalism. It is however striking that the structure can be examined very concretely, and that the abelian gerbe is embedded in the (non-abelian) $O(d,d)$.

We expect "slightly" non-abelian gerbes to arise in the U-duality context, as soon as the 6-form dual to the 3-form becomes important, *i.e.*, for $n \geq 5$. These (finite) transformations have yet to be constructed.

Back to the geometric origin of duality symmetries. To what extent can they be obtained as "generalised isometries"? There is a severe restriction, that is a result of the section condition. The situation is analogous in the $O(d,d)$ and $E_{n(n)}$ cases, I review the $O(d,d)$ situation for simplicity.

The solution to the section condition, forcing all fields to depend only on a subset of the coordinates, identified as “ordinary” space, is not changed by generalised diffeomorphisms. Transformations in $O(d,d)$ preserving an isotropic subspace do not fill out the entire $O(d,d)$, but only $GL(d) \ltimes \wedge^2 \mathbf{d}$.

In a basis with $X = (x, \tilde{x})$, where x are the physical coordinates, they take the form

$$\begin{bmatrix} m & \bullet \\ 0 & m^{-1} \end{bmatrix}$$

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This means that a large part of $O(d,d,\mathbb{Z})$ is excluded. Even the simple T-duality transformation interchanging momenta and string windings (or x and $\tilde{x}$) through “$R \leftrightarrow \frac{1}{R}$” cannot be obtained as a generalised diffeomorphism.

It also means that such transformations are not available as transition functions on overlaps, so that genuinely “non-geometric” solutions can not be constructed.

How can the situation be saved? It turns out that double diffeomorphisms can be formulated not only using the algebraic invariant $O(d,d)$ metric η_{MN}, but any pseudo-Riemannian (split-signature) metric $H_{MN}(X)$:

$$\mathscr{L}_\xi V_M = (L_\xi V)_M - H_{MP} H^{NQ} D_Q \xi^P V_N \,.$$

where now the covariant derivative D contains the torsion-free affine connection for H.

The potential curvature obstructions in the algebra then “miraculously” cancel [19, 23]. Some further flatness restrictions are imposed by the consistency of the covariant section condition. The defining metric H is not dynamical.

Corresponding statements should be true for exceptional groups, but there the structure in question is not a metric structure.

The point of such a “pre-geometric” formulation is that it becomes clear that any isometry of H will be a global symmetry of the model. For infinitesimal (continuous) isometries, the statement $[L_u, \mathscr{L}_\xi] = \mathscr{L}_{[u,\xi]}$ can be verified explicitly, and the analogous statement is true for finite isometries, *e.g.* the discrete isometries of a torus, where $O(d,d;\mathbb{Z})$ is the isometry subgroup of the mapping class group of the $2d$-torus.

Isometries are automorphisms of the generalised diffeomorphisms. Their parameters are not restricted by the section condition, and may therefore We may, and should, declare them as part of the gauge symmetry. By doing this, the problem of geometrising duality is solved, and a prescription is obtained for how fields transform under it. The same is true for non-geometric field configurations, where monodromy around some loop in a base space may take values in the gauge group, now containing T-duality transformations.

To conclude:

- Extended geometry unifies metric and tensor fields, and their respective gauge symmetries, in a framework providing interesting generalisations of ordinary geometry.
- Many questions remain to be examined, especially concerning finite transformations and the construction of "generalised manifolds". Most pressing is the issue of the section condition. Although it is covariant, its solutions explicitly break the continuous duality groups. It should preferably be promoted to a dynamically generated constraint, which would allow for true actions.
- A superspace formulation seems realistic for T-duality, but more problematic for U-duality. Simultaneous manifestation of supersymmetry and duality through some generalisation of pure spinor superfields may prove very powerful.
- Partial results exist on exceptional diffeomorphisms for $n > 7$. This is where dual gravity becomes important. This deserves further investigation.

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