Weakly curved background T-duals*

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Abstract

We discuss the generalized T-dualization procedure, its connection to the standard procedure, and the results of its application to the arbitrary set of coordinates of the closed string moving in the weakly curved background. This background consists of a constant metric and linearly coordinate dependent Kalb-Ramond field with infinitesimal strength. We unite all the results into a T-dualization diagram, representing all T-dual theories, the ways to obtain the theories from one another and the T-dual coordinate transformation laws connecting the corresponding coordinates.

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T-duality is a symmetry seen in the string spectrum. It was observed that for toroidal compactifications [1], where one dimension is compactified on a circle of radius R and the corresponding dual dimension is compactified on a circle of radius $1/R$, one obtains the discription of a string with the same physical properties. So, there exist different string theories, describing the string in the geometrically different backgrounds, with the same predictions. Such a symmetry is not present in any point particle theory [2, 3, 4, 5], and because of that the explanation for T-duality was sought for in a fact that strings can wrap around compactified dimensions. Investigation of T-duality lead to a discovery of a Buscher T-dualization procedure [6, 7], which gave a prescription how to find the T-dual theory for same known theory. The procedure is applicable along isometry directions, what allowed the investigation of a backgrounds which do not depend on some coordinates. This procedure enabled the investigation of the properties of the background connected by T-duality. It was discovered that geometric backgrounds transform to the non-geometric backgrounds and these to different non-geometric backgrounds, which differ in a form of the background fluxes, some of which are not locally well defined [8, 9]. T-duality is also investigated for the double string theories, where T-duality is a manifest symmetry [10, 11, 12, 13].

In this talk we will discuss the results of T-dualizations done for the closed string moving in the weakly curved background, using the generalized T-dualization procedure, defined in our paper [14]. This background depends on all the space-time coordinates and as such was not a candidate for T-dualization using the standard T-dualization procedures. In paper [14], we presented generalized T-dualization procedure applicable to all space-time directions regardless of the possible background coordinate dependence. We obtained the T-dual theory which is a result of T-dualizing all the initial coordinates. In paper [15], we broadened the investigation by considering T-dualization of an arbitrary set of coordinates of both initial and its completely T-dual theory. We will recapitulate the results here and discuss further investigations. We obtained the T-dualization diagram describing the relation between all string theories T-dual to the string moving in a weakly curved background, their backgrounds and giving the T-duality laws connecting the corresponding coordinates.

So, let us start by the action describing a closed string moving in a coordinate dependent background

$$S[x] = \kappa \int_\Sigma d^2\xi \; \partial_+ x^\mu \Pi_{+\mu\nu}(x) \partial_- x^\nu, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma \tag{1}$$

given in the conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$. The background field composition is defined by

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x). \tag{2}$$

It consists of a symmetric metric tensor $G_{\mu\nu} = G_{\nu\mu}$ and an antisymmetric Kalb-Ramond field $B_{\mu\nu} = -B_{\nu\mu}$. The background must obey the following space-time equations of motion

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} = 0, \quad D_\rho B^\rho{}_{\mu\nu} = 0, \tag{3}$$

in order to have a consistent quantum theory. We will consider one of the simplest coordinate dependent solutions, the weakly curved background, composed of a constant metric and linearly coordinate dependent Kalb-Ramond field which has an infinitesimal field strength

$$G_{\mu\nu}(x) = const,\ B_{\mu\nu}(x) = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho, \quad b_{\mu\nu}, B_{\mu\nu\rho} = const. \tag{4}$$

What are the backgrounds T-dual to this background? As, the standard T-dualization procedure is applicable to coordinate directions which do not appear as background field arguments, and the weakly curved background depends on all space-time coordinates, this procedure could not provide the answer to this question. So, a generalization of a T-dualization procedure which does not have this limitation had to be made. The main difference between the procedures obviously must be connected to background fields argument. We presented the new T-dualization procedure in [14].

Both procedures are built as a localization of a global coordinate shift symmetry $\delta x^\mu = \lambda^\mu = const$. One introduces the gauge fields v^μ_α and substitutes the ordinary derivatives with the covariant ones

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha. \tag{5}$$

Imposing the following transformation law for the gauge fields

$$\delta v^\mu_\alpha = -\partial_\alpha \lambda^\mu, \quad (\lambda^\mu = \lambda^\mu(\tau, \sigma)) \tag{6}$$

one obtains that $\delta D_\alpha x^\mu = 0$. If the background does not depend on the coordinates which are T-dualized the gauge invariant action is already obtained. But, what if the background depends on all the coordinates? The additional step must be introduced. It consists of a substitution of background field argument (the coordinate x^μ), by the invariant argument (invariant coordinate) defined as a line integral of the covariant derivatives of the original coordinate

$$\Delta x^\mu_{inv} \equiv \int_P d\xi^\alpha\, D_\alpha x^\mu = x^\mu - x^\mu(\xi_0) + \Delta V^\mu, \tag{7}$$

where

$$\Delta V^\mu \equiv \int_P d\xi^\alpha v^\mu_\alpha. \tag{8}$$

Consequently the arguments of the background fields will be nonlocal. Here, they are defined as the line integrals of the gauge fields, and as such are nonlocal. Later, once the explicit form of T-dual theories is obtained the non locality will appear as dependence on double coordinates.

In order to obtain the physically equivalent theories, one must make the introduced gauge fields nonphysical which is done by requiring that there field strength

$$F^{\mu}_{\alpha\beta} \equiv \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha \tag{9}$$

must be zero. This is achieved by adding the Lagrange multiplier y_μ term to the Lagrangian. Finally, the gauge invariant action, physically equivalent to the initial action is

$$S_{inv} = \kappa \int d^2\xi \Big[D_+ x^\mu \Pi_{+\mu\nu}(\Delta x_{inv}) D_- x^\nu + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu)\Big]. \tag{10}$$

Fixing the gauge $x^\mu(\xi) = x^\mu(\xi_0)$, one obtains

$$S_{fix}[y, v_\pm] = \kappa \int d^2\xi \Big[v^\mu_+ \Pi_{+\mu\nu}(\Delta V) v^\nu_- + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu)\Big]. \tag{11}$$

The gauge fixed action is the main crossway of the procedure, for an appropriate equation of motion it can transform both to initial action and to the T-dual action. For the equation of motion obtained varying the action over the Lagrange multipliers $\partial_+ v^\mu_- - \partial_- v^\mu_+ = 0$, with solution $v^\mu_\pm = \partial_\pm x^\mu$, the gauge fixed action reduces to the initial action. For the equation of motion obtained varying the action over the gauge fields $\Pi_{\mp\mu\nu}[\Delta V] v^\nu_\pm + \frac{1}{2}\partial_\pm y_\mu = \mp \beta^\mp_\mu[V]$, where $\beta^\alpha_\mu[V] \equiv \partial_\mu B_{\nu\rho} \epsilon^{\alpha\beta} V^\nu \partial_\beta V^\rho$, one obtains the T-dual theory. Comparing the solutions for the gauge fields in these two directions, one obtains the T-dual coordinate transformation laws. So, for application along all directions we obtain the following connection

$$S[x] = \kappa \int_\Sigma d^2\xi\, \partial_+ x^\mu \Pi_{+\mu\nu}(x) \partial_- x^\nu \Leftrightarrow {}^\star S[y] = \frac{\kappa^2}{2} \int d^2\xi\, \partial_+ y_\mu \Theta^{\mu\nu}_-(\Delta V) \partial_- y_\nu, \tag{12}$$

with $\Delta V^\mu = V^\mu(\xi) - V^\mu(\xi_0)$, $\quad V^\mu = (g^{-1})^{\mu\nu}[(2bG^{-1})_\nu{}^\rho y_\rho + \tilde{y}_\nu]$. The dual background field composition is defined by

$$\Theta^{\mu\nu}_\pm = -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}, \tag{13}$$

and consequently the T-dual background are

$$\begin{aligned} G_{\mu\nu} \quad &\Leftrightarrow \quad {}^\star G^{\mu\nu}(y,\tilde{y}) = (G_E^{-1})^{\mu\nu}(\Delta V), \\ B_{\mu\nu}(x) \quad &\Leftrightarrow \quad {}^\star B^{\mu\nu}(y,\tilde{y}) = \frac{\kappa}{2}\theta^{\mu\nu}(\Delta V), \end{aligned} \tag{14}$$

where $G_{E\mu\nu}$ and $\theta^{\mu\nu}$ are the effective metric and the noncommutativity parameter for open bosonic string, which are

$$G_{E\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu},\ \theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}. \tag{15}$$

But what if one does not consider T-dualization over all coordinates, but only some set of coordinates. To investigate this problem, let us mark a T-dualization along direction x^μ by T^μ and a T-dualization along dual direction y_μ by T_μ. Also mark the T-dualizations along some d initial directions, all other $D-d$ initial directions, and all initial directions by

$$\mathcal{T}^a = \circ_{n=1}^{d} T^{\mu_n}, \qquad \mathcal{T}^i = \circ_{n=d+1}^{D} T^{\mu_n}, \qquad \mathcal{T} = \circ_{n=1}^{D} T^{\mu_n} \tag{16}$$

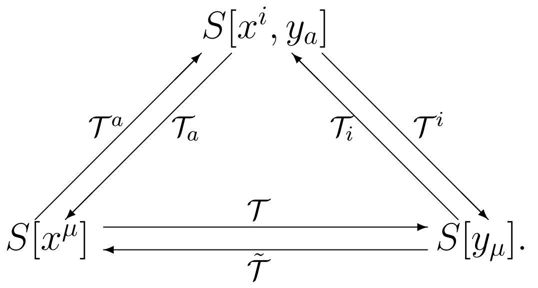
and T-dualizations along corresponding dual directions by

$$\mathcal{T}_a = \circ_{n=1}^{d} T_{\mu_n}, \qquad \mathcal{T}_i = \circ_{n=d+1}^{D} T_{\mu_n}, \qquad \widetilde{\mathcal{T}} = \circ_{n=1}^{D} T_{\mu_n} \tag{17}$$

$\mu_n \in (0, 1, \ldots, D-1)$. We showed in [15] that these T-dualizations form an Abelian group

$$\mathcal{T}^i \circ \mathcal{T}^a = \mathcal{T}, \quad \mathcal{T}_i \circ \mathcal{T}_a = \widetilde{\mathcal{T}}, \quad \mathcal{T}_a \circ \mathcal{T}^a = 1. \tag{18}$$

We showed that all the theories T-dual to the theory of the closed bosonic string are the part of the T-dualization diagram, given by



This diagram clearly describes the connection between arbitrary theory and the initial and completely T-dual theory. The explicit form of the theory obtained T-dualizing some set (marked by a) of the initial coordinates is the following

$$\begin{aligned} S[x^i, y_a] \;=\; \kappa \int d^2\xi \Big[& \partial_+ x^i \overline{\Pi}_{+ij}(x^i, \Delta V^a(x^i, y_a)) \partial_- x^j \\ & -\kappa\, \partial_+ x^i \Pi_{+ia}(x^i, \Delta V^a(x^i, y_a)) \tilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a)) \partial_- y_b \\ & +\kappa\, \partial_+ y_a \tilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a)) \Pi_{+bi}(x^i, \Delta V^a(x^i, y_a)) \partial_- x^i \\ & +\frac{\kappa}{2}\, \partial_+ y_a \tilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a)) \partial_- y_b \Big]. \end{aligned} \tag{19}$$

The new background field compositions $\overline{\Pi}_{\pm ij}$ and $\tilde{\Theta}_\pm^{ab}$ are defined as the inveres of the ordinary background field compositions $\Theta_\mp^{jk}$ and $\Pi_{\mp bc}$ reduced to the appropriate d and $D-d$ dimensional subspaces

$$\overline{\Pi}_{\pm ij} \Theta_\mp^{jk} = \Theta_\mp^{kj} \overline{\Pi}_{\pm ji} = \frac{1}{2\kappa} \delta_i^k, \tag{20}$$

$$\tilde{\Theta}^{ab}_{\pm}\Pi_{\mp bc} = \Pi_{\mp cb}\tilde{\Theta}^{ba}_{\pm} = \frac{1}{2\kappa}\delta^a_c. \tag{21}$$

It can be shown that

$$\overline{\Pi}_{+ij} \equiv \Pi_{+ij} - 2\kappa\Pi_{+ia}\tilde{\Theta}^{ab}_{-}\Pi_{+bj}. \tag{22}$$

The argument of the background fields is

$$\begin{aligned}\Delta V^{(0)a}(x^i, y_a) &= -\kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} + \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta x^{(0)i} \\ &- \kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} - \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta\tilde{x}^{(0)i} \\ &- \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} + \tilde{\Theta}^{ab}_{0-}\Big]\Delta y^{(0)}_b - \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} - \tilde{\Theta}^{ab}_{0-}\Big]\Delta\tilde{y}^{(0)}_b, \end{aligned}\tag{23}$$

where $\Delta x^\mu(\xi) = x^\mu(\xi) - x^\mu(\xi_0)$ and $\Delta y_\mu(\xi) = y_\mu(\xi) - y_\mu(\xi_0)$ while $\Delta\tilde{x}^\mu(\xi)$ and $\Delta\tilde{y}_\mu(\xi)$ are their duals, defined by

$$\Delta\tilde{x}^\mu(\xi) = \int_P d\xi^\alpha\, \varepsilon^\beta{}_\alpha\, \partial_\beta x^\mu, \quad \Delta\tilde{y}(\xi) = \int_P d\xi^\alpha\, \varepsilon^\beta{}_\alpha\, \partial_\beta y_\mu. \tag{24}$$

Calculating the symmetric and antisymmetric part of the background fields we obtain the T-dual metric and Kalb-Ramond field:

$$\begin{aligned}
&\bullet G_{ij} = \overline{G}_{ij} = G_{ij} - G_{ia}(\tilde{G}^{-1}_E)^{ab}G_{bj} \\
&\quad -2\kappa\Big(B_{ia}\tilde{\theta}^{ab}G_{bj} + G_{ia}\tilde{\theta}^{ab}B_{bj}\Big) - 4B_{ia}(\tilde{G}^{-1}_E)^{ab}B_{bj} \\
&\bullet B_{ij} = \overline{B}_{ij} = B_{ij} - \frac{\kappa}{2}G_{ia}\tilde{\theta}^{ab}G_{bj} - B_{ia}(\tilde{G}^{-1}_E)^{ab}G_{bj} \\
&\quad -G_{ia}(\tilde{G}^{-1}_E)^{ab}B_{bj} - 2\kappa B_{ia}\tilde{\theta}^{ab}B_{bj} \\
&\bullet G^{ab} = (\tilde{G}^{-1}_E)^{ab} \\
&\bullet B^{ab} = \frac{\kappa}{2}\tilde{\theta}^{ab} \\
&\bullet G^a{}_i = \kappa\tilde{\theta}^{ab}G_{bi} + 2(\tilde{G}^{-1}_E)^{ab}B_{bi} \\
&\bullet B^a{}_i = \kappa\tilde{\theta}^{ab}B_{bi} + \frac{1}{2}(\tilde{G}^{-1}_E)^{ab}G_{bi}.
\end{aligned}\tag{25}$$

As the constituents of the dual background field there appear the effective metric in the d-dimensional subspace a, defined by

$$\tilde{G}_{Eab} \equiv G_{ab} - 4B_{ac}(\tilde{G}^{-1})^{cd}B_{db}, \tag{26}$$

the non-commutativity parameter in the same subspace

$$\tilde{\theta}^{ab} \equiv -\frac{2}{\kappa}(\tilde{G}^{-1}_E)^{ac}B_{cd}(\tilde{G}^{-1})^{db}, \tag{27}$$

which combined give the new theta function $\tilde{\Theta}^{ab}_{\pm} = \tilde{\theta}^{ab} \mp \frac{1}{\kappa}(\tilde{G}^{-1}_E)^{ab}$.

Every arrow in the T-duality diagram is accompanied with the appropriate T-dual coordinate transformation law. These are obtained comparing the solutions for the gauge fields in a T-dualization procedures performed between two actions in both directions. The laws for transitions

$$\mathcal{T}^a : S[x^\mu] \to S[x^i, y_a], \quad \mathcal{T}_a : S[x^i, y_a] \to S[x^\mu],$$

which are inverse to each other, are given by

$$\begin{aligned} \partial_\mp x^a &\cong -2\kappa \tilde{\Theta}^{ab}_{\mp}(x^i, \Delta V^a(x^i, y_a)) \cdot \\ &\cdot \Big[\Pi_{\pm bi}(x^i, \Delta V^a(x^i, y_a))\partial_\mp x^i + \frac{1}{2}\partial_\mp y_b \mp \beta^{\pm}_b(x^i, V^a(x^i, y_a))\Big] \\ x^{(0)a} &\cong V^{(0)a}(x^i, y_a) \end{aligned} \tag{28}$$

and its inverse

$$\begin{aligned} \partial_\mp y_a &\cong -2\Pi_{\pm a\mu}(x)\partial_\mp x^\mu \pm 2\beta^{\pm}_a(x), \\ y^{(0)}_a &\cong U^{(0)}_a(x). \end{aligned} \tag{29}$$

These relations enable an investigation of the closed string non-commutativity and other geometric properties of the T-dual backgrounds. One can determine the geometric structure for an arbitrary sigma model in a T-duality diagram, find the connection between the Poisson structures of T-dual theories and the relations between non-commutativity parameters. The coordinates of the closed string are commutative when the string moves in a constant background. In a three dimensional space with the Kalb-Ramond field depending on one of the coordinates, successive T-dualizations along isometry directions lead to a theory with Q flux and the non-commutative coordinates [16, 17, 18]. Using the generalized T-dualization procedure, we found the non-commutativity characteristics of a closed string moving in the weakly curved background [15] comparing the initial and completely T-daul theory. One can expect the further investigations will reveal novelties regarding the form of the fluxes of all T-dual background forming a diagram. For now it is known that all fluxes are of type R.

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