# Derivation of the trigonometric Gaudin Hamiltonians*

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Abstract

Following Sklyanin's proposal in the rational case, we derive the generating function of the Gaudin Hamiltonians in the trigonometric case. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the inhomogeneous XXZ Heisenberg spin chain and the central element, the so-called Sklyanin determinant. The corresponding Gaudin Hamiltonians are obtained as the residues of the generating function.

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## 1. Introduction

Gaudin models were introduced as interacting spins in a chain [1, 2, 3, 4]. In this approach, these models were obtained as a quasi-classical limit of the integrable quantum chains. Moreover, the Gaudin models were extended to any simple Lie algebra, with arbitrary irreducible representation at each site of the chain [4].

The rational $s\ell(2)$ invariant model was studied in the framework of the quantum inverse scattering method [5]. In his studies, Sklyanin used the $s\ell(2)$ invariant classical r-matrix [5]. A generalization of these results to all cases when skew-symmetric r-matrix satisfies the classical Yang-Baxter equation [6] was relatively straightforward [7, 8]. Therefore, considerable

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attention has been devoted to Gaudin models corresponding to the the classical r-matrices of simple Lie algebras [9, 10] and Lie superalgebras[11, 12]. In the case of the $s\ell(2)$ Gaudin system, its relation to Knizhnik-Zamolodchikov equation of conformal field theory [13, 14, 15] or the method of Gauss factorization [16], provided alternative approaches to computation of correlation functions. The non-unitary r-matrices and the corresponding Gaudin models have been studied recently, see [17, 18] and the references therein. In [19] we have derived the generating function of the $s\ell(2)$ Gaudin Hamiltonians with boundary terms. Moreover, we have implemented the algebraic Bethe ansatz, based on the appropriate non-unitary r-matrices and the corresponding linear bracket, obtaining the spectrum of the generating function and the corresponding Bethe equations [19].

Here, following Sklyanin's proposal in the rational case [5, 19], we derive the generating function of the Gaudin Hamiltonians in the trigonometric case. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the inhomogeneous XXZ Heisenberg spin chain and the central element, the so-called quantum determinant.

## 2. Inhomogeneous XXZ Heisenberg spin chain

With the aim deriving the Gaudin Hamiltonians in the trigonometric case, we consider the R-matrix of the XXZ Heisenberg spin chain [20, 21, 22]

$$R(\lambda,\eta)=\begin{pmatrix}\sinh(\lambda+\eta) & 0 & 0 & 0\\ 0 & \sinh(\lambda) & \sinh(\eta) & 0\\ 0 & \sinh(\eta) & \sinh(\lambda) & 0\\ 0 & 0 & 0 & \sinh(\lambda+\eta)\end{pmatrix}. \tag{1}$$

This R-matrix satisfies the Yang-Baxter equation in the space $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ [24, 23]

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu)=R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu). \tag{2}$$

Here we study the inhomogeneous XXZ spin chain with $N$ sites, characterised by the local space $V_m=\mathbb{C}^{2s+1}$ and an inhomogeneous parameter $\alpha_m$. The Hilbert space of the system is

$$\mathcal{H}=\overset{N}{\underset{m=1}{\otimes}}V_m=(\mathbb{C}^{2s+1})^{\otimes N}. \tag{3}$$

We introduce the Lax operator as the following two-by-two matrix in the auxiliary space $V_0=\mathbb{C}^2$,

$$\mathbb{L}_{0m}(\lambda)=\frac{1}{\sinh(\lambda)}\begin{pmatrix}\sinh\left(\lambda\mathbb{1}_m+\eta S^3_m\right) & \sinh(\eta)S^-_m\\ \sinh(\eta)S^+_m & \sinh\left(\lambda\mathbb{1}_m-\eta S^3_m\right)\end{pmatrix}. \tag{4}$$

When the quantum space is also a spin $\frac{1}{2}$ representation, the Lax operator becomes the $R$-matrix,

$$\mathbb{L}_{0m}(\lambda) = \frac{1}{\sinh(\lambda)} R_{0m}\left(\lambda - \eta/2\right).$$

Due to the commutation relations (34), it is straightforward to check that the Lax operator satisfies the RLL-relations

$$R_{00'}(\lambda-\mu)\mathbb{L}_{0m}(\lambda-\alpha_m)\mathbb{L}_{0'm}(\mu-\alpha_m) = \mathbb{L}_{0'm}(\mu-\alpha_m)\mathbb{L}_{0m}(\lambda-\alpha_m)R_{00'}(\lambda-\mu). \tag{5}$$

The so-called monodromy matrix

$$T(\lambda) = \mathbb{L}_{0N}(\lambda - \alpha_N) \cdots \mathbb{L}_{01}(\lambda - \alpha_1) \tag{6}$$

is used to describe the system. For simplicity we have omitted the dependence on the quasi-classical parameter $\eta$ and the inhomogeneous parameters $\{\alpha_j, j = 1, \ldots, N\}$. Notice that $T(\lambda)$ is a two-by-two matrix acting in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{7}$$

From RLL-relations (5) it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda - \mu)T_0(\lambda)T_{0'}(\mu) = T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda - \mu). \tag{8}$$

The periodic boundary conditions and the RTT-relations (8) imply that the transfer matrix

$$t(\lambda) = \mathrm{tr}_0 T(\lambda), \tag{9}$$

commute at different values of the spectral parameter,

$$[t(\mu), t(\nu)] = 0. \tag{10}$$

The RTT-relations (8) admit a central element [5]

$$\Delta\left[T(\lambda)\right] = \mathrm{tr}_{00'} P^-_{00'} T_0\left(\lambda - \eta/2\right) T_{0'}\left(\lambda + \eta/2\right), \tag{11}$$

where

$$P^-_{00'} = \frac{-1}{2\sinh(\eta)} R_{00'}(-\eta) = \frac{\mathbb{1} - \mathcal{P}_{00'}}{2}, \tag{12}$$

where $\mathbb{1}$ is the identity and $\mathcal{P}$ is the permutation in $\mathbb{C}^2 \otimes \mathbb{C}^2$. A straightforward calculation shows that $\Delta\left[T(\lambda)\right]$ is a scalar operator

$$\Delta\left[T(\lambda)\right] = \prod_{m=1}^{N} \frac{\sinh\left(\lambda - \alpha_m + \frac{(2s_m+1)\eta}{2}\right) \sinh\left(\lambda - \alpha_m - \frac{(2s_m+1)\eta}{2}\right)}{\sinh\left(\lambda - \alpha_m + \frac{\eta}{2}\right) \sinh\left(\lambda - \alpha_m - \frac{\eta}{2}\right)}, \tag{13}$$

and therefore, it is evidently central,

$$\Big[\Delta\left[T(\lambda)\right]\,,\,T(\nu)\Big] = 0. \tag{14}$$

In the next section we will seek a linear combination of the transfer matrix (9) and the central element (11) whose quasi-classical expansion yields the generating function of the trigonometric Gaudin Hamiltonians in the case when the periodic boundary conditions are imposed.

## 3. Trigonometric Gaudin model

The Gaudin models were introduce as a quasi-classical limit of the integrable quantum chains [3, 4]. Therefore it is to be expected that the generating function of the trigonometric Gaudin Hamiltonians could be obtained from the quasi-classical expansion of the transfer matrix of the periodic XXZ Heisenberg spin chain. Thus, our first step is to consider the expansion of the monodromy matrix (6) with respect to the quasi-classical parameter $\eta$

$$\begin{aligned}
T(\lambda) &= \mathbb{1} + \eta \sum_{m=1}^{N} \frac{\sigma_0^3 \otimes \cosh(\lambda-\alpha_m) S_m^3 + \frac{1}{2}\left(\sigma_0^+ \otimes S_m^- + \sigma_0^- \otimes S_m^+\right)}{\sinh(\lambda-\alpha_m)} + \frac{\eta^2}{2} \mathbb{1}_0 \otimes \sum_{m=1}^{N} \left(S_m^3\right)^2 \\
&+ \frac{\eta^2}{2} \sum_{\substack{n,m=1\\ n\neq m}}^{N} \frac{\mathbb{1}_0 \otimes \left(\cosh(\lambda-\alpha_m)\cosh(\lambda-\alpha_n)\, S_m^3 S_n^3 + \frac{1}{2}\left(S_m^+ S_n^- + S_m^- S_n^+\right)\right)}{\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \\
&+ \frac{\eta^2}{2} \sum_{m=1}^{N} \sum_{n<m}^{N} \frac{\sigma_0^3 \otimes \left(S_m^- S_n^+ - S_m^+ S_n^-\right) + \sigma_0^+ \otimes \left(\cosh(\lambda-\alpha_m) S_m^3 S_n^- - \cosh(\lambda-\alpha_n) S_m^- S_n^3\right)}{2\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \\
&+ \frac{\eta^2}{2} \sum_{m=1}^{N} \sum_{n<m}^{N} \frac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_n) S_m^+ S_n^3 - \cosh(\lambda-\alpha_m) S_m^3 S_n^-\right)}{2\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \\
&+ \frac{\eta^2}{2} \sum_{m=1}^{N} \sum_{n>m}^{N} \frac{\sigma_0^3 \otimes \left(S_n^- S_m^+ - S_n^+ S_m^-\right) + \sigma_0^+ \otimes \left(\cosh(\lambda-\alpha_n) S_n^3 S_m^- - \cosh(\lambda-\alpha_m) S_n^- S_m^3\right)}{2\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_m)} \\
&+ \frac{\eta^2}{2} \sum_{m=1}^{N} \sum_{n>m}^{N} \frac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_m) S_n^+ S_m^3 - \cosh(\lambda-\alpha_n) S_n^3 S_m^-\right)}{2\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_m)} + \mathcal{O}(\eta^3).
\end{aligned} \tag{15}$$

It is important to notice that the spin operators $S_m^\alpha$, with $\alpha = +, -, 3$, on the right hand side of (15) satisfy the usual commutation relations

$$[S_m^3, S_n^\pm] = \pm S_m^\pm\, \delta_{mn}, \quad [S_m^+, S_n^-] = 2 S_m^3\, \delta_{mn}. \tag{16}$$

If the Gaudin Lax matrix is defined by

$$L_0(\lambda) = \sum_{m=1}^{N} \frac{\sigma_0^3 \otimes \cosh(\lambda-\alpha_m) S_m^3 + \frac{1}{2}\left(\sigma_0^+ \otimes S_m^- + \sigma_0^- \otimes S_m^+\right)}{\sinh(\lambda-\alpha_m)} \tag{17}$$

and the quasi-classical property of the R-matrix (1) [23]

$$\frac{1}{\sinh(\lambda)} R(\lambda) = \mathbb{1} - \eta r(\lambda) + \mathcal{O}(\eta^2), \tag{18}$$

where

$$r(\lambda) = \frac{-1}{2\sinh(\lambda)} \left( \cosh(\lambda)(\mathbb{1} \otimes \mathbb{1} + \sigma^3 \otimes \sigma^3) + \frac{1}{2} \left( \sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+ \right) \right), \tag{19}$$

is taken into account, then substitution of the expansion (15) into the RTT-relations (8) yields the so-called Sklyanin linear bracket [5]

$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)]\,. \tag{20}$$

The classical r-matrix (19) has the unitarity property

$$r_{21}(-\lambda) = -r_{12}(\lambda), \tag{21}$$

and satisfies the classical Yang-Baxter equation [6]

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda - \mu), r_{13}(\lambda) + r_{23}(\mu)] = 0. \tag{22}$$

Thus the Sklyanin linear bracket (20) is anti-symmetric and it obeys the Jacobi identity. It follows that the entries of the Lax matrix (17) generate a Lie algebra relevant for the Gaudin model.

Using the expansion (15) it is evident that

$$\begin{aligned} t(\lambda) = 2 + \eta^2 \sum_{m=1}^{N} \Bigg( & \left(S_m^3\right)^2 \\ & + \textstyle\sum_{n\neq m}^{N} \frac{\cosh(\lambda-\alpha_m)\cosh(\lambda-\alpha_n)\, S_m^3 S_n^3 + \frac{1}{2}\left(S_m^+ S_n^- + S_m^- S_n^+\right)}{\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \Bigg) + \mathcal{O}(\eta^3). \end{aligned} \tag{23}$$

Analogously, we can expand (11) to obtain

$$
\begin{aligned}
&\Delta\left[T(\lambda)\right] = \mathbb{1} + \eta\, \mathrm{tr} L(\lambda) + \tfrac{\eta^2}{2}\left(\mathrm{tr}^2 L(\lambda) - \mathrm{tr}\, L^2(\lambda)\right)\\
&+\eta^2 \textstyle\sum_{m=1}^N \left( \left(S_m^3\right)^2 + \sum_{n\neq m}^N \frac{\cosh(\lambda-\alpha_m)\cosh(\lambda-\alpha_n)\, S_m^3 S_n^3 + \frac12\left(S_m^+ S_n^- + S_m^- S_n^+\right)}{\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)}\right)\\
&+\tfrac{\eta^2}{2}\mathrm{tr}_{00'} P^-_{00'} \textstyle\sum_{m=1}^N \sum_{m>n}^N \left( \frac{\sigma_0^3 \otimes \left(S_m^- S_n^+ - S_m^+ S_n^+\right) + \sigma_0^+ \otimes \left(\cosh(\lambda-\alpha_m) S_m^3 S_n^- - \cosh(\lambda-\alpha_n) S_m^- S_n^3\right)}{2\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)}\right.\\
&\left.+\tfrac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_n) S_m^+ S_n^3 - \cosh(\lambda-\alpha_m) S_m^3 S_n^+\right)}{2\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)}\right)\mathbb{1}_{0'}\\
&+\tfrac{\eta^2}{2}\mathrm{tr}_{00'} P^-_{00'} \textstyle\sum_{m=1}^N \sum_{n>m}^N \left( \frac{\sigma_0^3 \otimes \left(S_n^- S_m^+ - S_n^+ S_m^+\right) + \sigma_0^+ \otimes \left(\cosh(\lambda-\alpha_n) S_n^3 S_m^- - \cosh(\lambda-\alpha_m) S_n^- S_m^3\right)}{2\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_m)}\right.\\
&\left.+\tfrac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_m) S_n^+ S_m^3 - \cosh(\lambda-\alpha_n) S_n^3 S_m^+\right)}{2\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_n)}\right)\mathbb{1}_{0'}\\
&+\tfrac{\eta^2}{2}\mathrm{tr}_{00'} P^-_{00'} \mathbb{1}_0 \textstyle\sum_{m=1}^N \sum_{m>n}^N \left( \frac{\sigma_{0'}^3 \otimes \left(S_m^- S_n^+ - S_m^+ S_n^+\right) + \sigma_{0'}^+ \otimes \left(\cosh(\lambda-\alpha_m) S_m^3 S_n^- - \cosh(\lambda-\alpha_n) S_m^- S_n^3\right)}{2\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)}\right.\\
&\left.+\tfrac{\sigma_{0'}^- \otimes \left(\cosh(\lambda-\alpha_n) S_m^+ S_n^3 - \cosh(\lambda-\alpha_m) S_m^3 S_n^+\right)}{2\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)}\right)\\
&+\tfrac{\eta^2}{2}\mathrm{tr}_{00'} P^-_{00'} \mathbb{1}_0 \textstyle\sum_{m=1}^N \sum_{n>m}^N \left( \frac{\sigma_{0'}^3 \otimes \left(S_n^- S_m^+ - S_n^+ S_m^+\right) + \sigma_{0'}^+ \otimes \left(\cosh(\lambda-\alpha_n) S_n^3 S_m^- - \cosh(\lambda-\alpha_m) S_n^- S_m^3\right)}{2\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_m)}\right.\\
&\left.+\tfrac{\sigma_{0'}^- \otimes \left(\cosh(\lambda-\alpha_m) S_n^+ S_m^3 - \cosh(\lambda-\alpha_n) S_n^3 S_m^+\right)}{2\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_n)}\right) + \mathcal{O}(\eta^3),
\end{aligned}
\tag{24}
$$

where $L(\lambda)$ is given in (17). The final expression for the expansion of $\Delta\left[T(\lambda)\right]$ is obtained after taking all the traces

$$
\begin{aligned}
&\Delta\left[T(\lambda)\right] = \mathbb{1} + \eta^2\\
&\times \sum_{m=1}^N \left( \left(S_m^3\right)^2 + \sum_{n\neq m}^N \frac{\cosh(\lambda-\alpha_m)\cosh(\lambda-\alpha_n)\, S_m^3 S_n^3 + \frac12\left(S_m^+ S_n^- + S_m^- S_n^+\right)}{\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)}\right)\\
&-\frac{\eta^2}{2}\mathrm{tr}\, L^2(\lambda) + \mathcal{O}(\eta^3).
\end{aligned}
\tag{25}
$$

To obtain the generation function of the Gaudin Hamiltonians notice that (23) and (25) yield

$$
t(\lambda) - \Delta\left[T(\lambda)\right] = \mathbb{1} + \frac{\eta^2}{2}\mathrm{tr}\, L^2(\lambda) + \mathcal{O}(\eta^3). \tag{26}
$$

Therefore

$$
\tau(\lambda) = \frac12 \mathrm{tr}\, L^2(\lambda) \tag{27}
$$

commute for different values of the spectral parameter,

$$
\left[\tau(\lambda), \tau(\mu)\right] = 0. \tag{28}
$$

Moreover, substituting (17) into (27) it is straightforward to obtain the expansion

$$\tau(\lambda) = \sum_{m=1}^{N} \frac{s_m(s_m+1)}{\sinh^2(\lambda - \alpha_m)} + 2\sum_{m=1}^{N} \coth(\lambda - \alpha_m)\, H_m + (S_{gl}^3)^2\,, \tag{29}$$

with the Gaudin Hamiltonians

$$H_m = \sum_{n\neq m}^{N} \coth(\alpha_m - \alpha_n)\, S_m^3 S_n^3 + \frac{1}{2\sinh(\alpha_m - \alpha_n)} \left(S_m^+ S_n^- + S_m^- S_n^+\right) \tag{30}$$

and the global generator

$$S_{gl}^3 = \sum_{m=1}^{N} S_m^3. \tag{31}$$

The global generator defined above generates the $U(1)$ symmetry

$$\left[S_{gl}^3, H_m\right] = 0, \quad \text{with} \quad m = 1, 2 \ldots N. \tag{32}$$

Evidently, we have

$$[H_m, H_n] = 0, \quad \text{with} \quad m, n = 1, 2 \ldots N. \tag{33}$$

This shows that $\tau(\lambda)$ is the generating function of Gaudin Hamiltonians (30) when the periodic boundary conditions are imposed [5].

## 4. Conclusion

Following Sklyanin's proposal [5, 19], we have derive the generating function of the Gaudin Hamiltonians in the trigonometric case by considering the quasi-classical expansion of the linear combination of the transfer matrix of the XXZ Heisenberg spin chain and the corresponding quantum determinant. The Gaudin Hamiltonians are obtained as the residues of the generating function. It would be of considerable interest to generalise these results to the case of non-periodic boundary conditions.

## A Basic definitions

We consider the operators $S^\alpha$ with $\alpha = +, -, 3$, acting in some (spin $s$) representation space $\mathbb{C}^{2s+1}$ with the commutation relations [25]

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = \frac{\sinh(2\eta S^3)}{\sinh(\eta)} = [2S^3]_q, \tag{34}$$

with $q = e^{\eta}$. In the space $\mathbb{C}^{2s+1}$ these operators admit the following matrix representation [25, 26, 27]

$$S^3 = \sum_{i=1}^{2s+1} a_i e_{i\,i}, \quad S^+ = \sum_{i=1}^{2s+1} b_i e_{i\,i+1}, \quad \text{and} \quad S^- = \sum_{i=1}^{2s+1} b_i e_{i+1\,i} \tag{35}$$

where

$$(e_{ij})_{kl} = \delta_{i\,k}\delta_{j\,l}, \quad a_i = s+1-i, \quad b_i = \sqrt{[i]_q\ [2s+1-i]_q} \quad \text{and} \quad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{36}$$

In the particular case of spin $\frac{1}{2}$ representation, one recovers the Pauli matrices

$$S^{\alpha} = \frac{1}{2}\sigma^{\alpha} = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix}.$$

We consider a spin chain with N sites with spin $s$ representations, i.e. a local $\mathbb{C}^{2s+1}$ space at each site and the operators

$$S_m^{\alpha} = \mathbb{1} \otimes \cdots \otimes \underbrace{S^{\alpha}}_{m} \otimes \cdots \otimes \mathbb{1}, \tag{37}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$.

## References


[1] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press, 1987.

[2] E. S. Fradkin and A. A. Tseytlin, *Phys.Lett.* **B 158** (1985) 316; *Nucl.Phys.* **B 261** (1985) 1; C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, *Nucl.Phys.* **B 262** (1985) 593; T. Banks, D. Nemeschansky and A. Sen, *Nucl.Phys.* **B 277** (1986) 67; A. A. Tseytlin, *Int. J. Mod. Phys.* **A 4** (1989 1257; A. A. Tseytlin, *Int. J. Mod. Phys.* **A 4** (1989 1257.

[3] M. Gaudin, *Diagonalisation d'une classe d'hamiltoniens de spin*, J. Physique **37** (1976) 1087–1098.

[4] M. Gaudin, *La fonction d'onde de Bethe*, chapter 13 Masson, Paris, 1983.

[5] E. K. Sklyanin, *Separation of variables in the Gaudin model*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **164** (1987) 151–169; translation in Journal of Soviet Mathematics Volume **47**, Issue 2 (1989) 2473–2488.

[6] A. A. Belavin and V. G. Drinfeld. *Solutions of the classical Yang-Baxter equation for simple Lie algebras* (in Russian), Funktsional. Anal. i Prilozhen. **16** (1982), no. 3, 1–29; translation in Funct. Anal. Appl. **16** (1982) no. 3, 159-180.

[7] E. K. Sklyanin and T. Takebe, *Algebraic Bethe ansatz for the XYZ Gaudin model*, Phys. Lett. **A 219** (1996) 217-225.

[8] M. A. Semenov-Tian-Shansky, *Quantum and classical integrable systems*, in Integrability of Nonlinear Systems, Lecture Notes in Physics **495** (1997) 314-377.

[9] B. Jurčo, *Classical Yang-Baxter equations and quantum integrable systems*, J. Math. Phys. **30** (1989) 1289–1293.

[10] B. Jurčo, *Classical Yang-Baxter equations and quantum integrable systems (Gaudin models)*, in Quantum groups (Clausthal, 1989), Lecture Notes in Phys. **370** (1990) 219–227.

[11] P. P. Kulish and N. Manojlović, *Creation operators and Bethe vectors of the* $osp(1|2)$ *Gaudin model*, J. Math. Phys. **42** no. 10 (2001) 4757–4778.

[12] P. P. Kulish and N. Manojlović, *Trigonometric* $osp(1|2)$ *Gaudin model*, J. Math.Phys. **44** no. 2 (2003) 676–700.

[13] H. M. Babujian and R. Flume, *Off-shell Bethe ansatz equation for Gaudin magnets and solutions of Knizhnik-Zamolodchikov equations*, Mod. Phys. Lett. **A 9** (1994) 2029–2039.

[14] B. Feigin, E. Frenkel, and N. Reshetikhin, *Gaudin model, Bethe ansatz and correlation functions at the critical level*, Commun. Math. Phys. **166** (1994) 27–62.

[15] N. Reshetikhin and A. Varchenko, *Quasiclassical asymptotics of solutions to the KZ equations*, in Geometry, Topology and Physics, Conf. Proc. Lecture Notes Geom., pages 293–273. Internat. Press, Cambridge, MA, 1995.

[16] E. K. Sklyanin, *Generating function of correlators in the* $sl_2$ *Gaudin model*, Lett. Math. Phys. **47** (1999) 275–292.

[17] T. Skrypnyk, *Non-skew-symmetric classical r-matrix, algebraic Bethe ansatz, and Bardeen-Cooper-Schrieffer-type integrable systems*, J. Math. Phys. **50** (2009) 033540, 28 pages.

[18] T. Skrypnyk, *"Z2-graded" Gaudin models and analytical Bethe ansatz*, Nuclear Physics **B 870** (2013), no. 3, 495–529.

[19] N. Cirilo António, N. Manojlović, E. Ragoucy and I. Salom, *Algebraic Bethe ansatz for the* $s\ell(2)$ *Gaudin model with boundary*, Nuclear Physics **B 893** (2015) 305-331; `arXiv:1412.1396`.

[20] N. Cirilo António, N. Manojlović and Z. Nagy, *Trigonometric* $s\ell(2)$ *Gaudin model with boundary terms*, Reviews in Mathematical Physics **25** No. 10 (2013) 1343004 (14 pages); `arXiv:1303.2481`.

[21] R. J. Baxter, *Partition function of the Eight-Vertex lattice model*, Annals of Physics **70**, Issue 1, March 1972, Pages 193-228.

[22] L. A. Takhtajan and L. D. Faddeev, *The quantum method for the inverse problem and the* $XYZ$ *Heisenberg model*, (in Russian) Uspekhi Mat. Nauk **34** No. 5 (1979) 13–63; translation in Russian Math. Surveys **34** No.5 (1979) 11–68.

[23] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method. Recent developments*, Lect. Notes Phys. **151** (1982), 61–119.

[24] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London (1982).

[25] P. P. Kulish and N. Yu. Reshetikhin, *Quantum linear problem for the sine-Gordon equation and higher representations*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **101** (1981) 101–110, translation in Journal of Soviet Mathematics **23**, Issue 4 (1983) Pages 2435-2441.

[26] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *Quantum groups*, Braid group, knot theory and statistical mechanics, 97–110, Adv. Ser. Math. Phys., 9, World Sci. Publ., Teaneck, NJ, 1989.

[27] A. Doikou, *A note on the boundary spin s XXZ chain*, Physics Letters **A 366** (2007) 556-562.