Creation operators of the non-periodic $s\ell(2)$ Gaudin model*

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Abstract

We define new creation operators relevant for implementation of the algebraic Bethe ansatz for the $s\ell(2)$ Gaudin model with the general reflection matrix. This approach is based on the linear bracket corresponding to the relevant non-unitary classical r-matrix.
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1. Introduction

In [1] we have derived the generating function of the $s\ell(2)$ Gaudin Hamiltonians with boundary terms. We have shown that the implementation of the algebraic Bethe ansatz requires an appropriate non-unitary r-matrices and the corresponding linear bracket [1]. The non-unitary r-matrices and the corresponding Gaudin models have been studied recently, see [2, 3] and the references therein. In [1] we have obtained the spectrum of the generating function and the corresponding Bethe equations. However, explicit and compact form of the Bethe vector $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, remained open. Our aim here is to propose creation operators which should solve this problem.

2. $s\ell(2)$ Gaudin model with boundary terms

The classical r-matrix relevant for the $s\ell(2)$ Gaudin model is given by [4]

$$r(\lambda) = -\frac{\mathcal{P}}{\lambda}, \tag{1}$$

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where $\mathcal{P}$ is the permutation matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$. This classical r-matrix satisfies the classical Yang-Baxter equation

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda-\mu), r_{13}(\lambda) + r_{23}(\mu)] = 0, \tag{2}$$

and it has the unitarity property

$$r_{21}(-\lambda) = -r_{12}(\lambda). \tag{3}$$

The general solution of the corresponding classical reflection equation [5, 6, 7]:

$$\begin{aligned} r_{12}(\lambda-\mu)K_1(\lambda)K_2(\mu) + K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu) = \\ = K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda) + K_2(\mu)K_1(\lambda)r_{21}(\lambda-\mu), \end{aligned} \tag{4}$$

is give by [1]

$$\widetilde{K}(\lambda) = \begin{pmatrix} \xi - \lambda & \widetilde{\psi}\lambda \\ \widetilde{\phi}\lambda & \xi + \lambda \end{pmatrix}. \tag{5}$$

An important preliminary step in the implementation of the algebraic Bethe ansatz for the open Gaudin model is to bring the K-matrix (5) to the upper, or lower, triangular form [1]

$$K(\lambda) = U^{-1}\widetilde{K}(\lambda)U = \begin{pmatrix} \xi - \lambda\nu & \lambda\psi \\ 0 & \xi + \lambda\nu \end{pmatrix}, \tag{6}$$

where $\psi = \widetilde{\phi} + \widetilde{\psi}$ and

$$U = \begin{pmatrix} -1-\nu & \widetilde{\phi} \\ \widetilde{\phi} & -1-\nu \end{pmatrix}, \tag{7}$$

with $\nu = \sqrt{1 + \widetilde{\phi}\,\widetilde{\psi}}$.

Here we study the $s\ell(2)$ Gaudin model with N sites, characterised by the local space $V_m = \mathbb{C}^{2s+1}$ and inhomogeneous parameter α_m. The Hilbert space of the system is

$$\mathcal{H} = \mathop{\otimes}_{m=1}^{N} V_m = (\mathbb{C}^{2s+1})^{\otimes N}. \tag{8}$$

Following [1] we introduce the Lax operator

$$\mathcal{L}_0(\lambda) = \begin{pmatrix} H(\lambda) & F(\lambda) \\ E(\lambda) & -H(\lambda) \end{pmatrix} = \sum_{m=1}^{N} \left(\frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} + \frac{K_0(\lambda)\vec{\sigma}_0 K_0^{-1}(\lambda) \cdot \vec{S}_m}{\lambda + \alpha_m} \right), \tag{9}$$

with the following local realisation for the entries of the Lax matrix

$$E(\lambda)=\sum_{m=1}^{N}\left(\frac{S_m^+}{\lambda-\alpha_m}+\frac{(\xi+\lambda\nu)S_m^+}{(\xi-\lambda\nu)(\lambda+\alpha_m)}\right), \tag{10}$$

$$F(\lambda)=\sum_{m=1}^{N}\left(\frac{S_m^-}{\lambda-\alpha_m}+\frac{(\xi-\lambda\nu)^2S_m^- -\lambda^2\psi^2S_m^+ -2\lambda\psi(\xi-\lambda\nu)S_m^3}{(\xi+\lambda\nu)(\xi-\lambda\nu)(\lambda+\alpha_m)}\right), \tag{11}$$

$$H(\lambda)=\sum_{m=1}^{N}\left(\frac{S_m^3}{\lambda-\alpha_m}+\frac{\lambda\psi\, S_m^+ +(\xi-\lambda\nu)S_m^3}{(\xi-\lambda\nu)(\lambda+\alpha_m)}\right). \tag{12}$$

Due to the commutation relations (36), it is straightforward to check that the Lax operator (9) satisfies the following linear bracket relations

$$[\mathcal{L}_0(\lambda),\mathcal{L}_{0'}(\mu)]=\left[r^K_{00'}(\lambda,\mu),\mathcal{L}_0(\lambda)\right]-\left[r^K_{0'0}(\mu,\lambda),\mathcal{L}_{0'}(\mu)\right], \tag{13}$$

where the non-unitary r-matrix is give by

$$r^K_{00'}(\lambda,\mu)=r_{00'}(\lambda-\mu)-K_{0'}(\mu)r_{00'}(\lambda+\mu)K^{-1}_{0'}(\mu). \tag{14}$$

The commutator (13) is obviously anti-symmetric. It obeys the Jacobi identity because the r-matrix (14) satisfies the classical Yang-Baxter equation

$$[r^K_{32}(\lambda_3,\lambda_2),r^K_{13}(\lambda_1,\lambda_3)]+[r^K_{12}(\lambda_1,\lambda_2),r^K_{13}(\lambda_1,\lambda_3)+r^K_{23}(\lambda_2,\lambda_3)]=0. \tag{15}$$

The linear bracket (13) based on the r-matrix $r^K_{00'}(\lambda,\mu)$ (14), corresponding to (6) and the classical r-matrix (1), defines the Lie algebra relevant for the open $s\ell(2)$ Gaudin model.

As it was shown in [1], it is instructive to introduce the new generators $e(\lambda), h(\lambda)$ and $f(\lambda)$ as the following linear combinations of the original ones

$$e(\lambda)=\tfrac{-\xi+\lambda\nu}{\lambda}E(\lambda),\quad h(\lambda)=\tfrac{1}{\lambda}\left(H(\lambda)-\tfrac{\psi\lambda}{2\xi}\,E(\lambda)\right),$$
$$f(\lambda)=\tfrac{1}{\lambda}\left((\xi+\lambda\nu)F(\lambda)+\psi\lambda H(\lambda)\right). \tag{16}$$

The key observation is that in the new basis we have

$$[e(\lambda),e(\mu)]=[h(\lambda),h(\mu)]=[f(\lambda),f(\mu)]=0. \tag{17}$$

Therefore there are only three nontrivial relations

$$[h(\lambda), e(\mu)] = \frac{2}{\lambda^2 - \mu^2} \left(e(\mu) - e(\lambda) \right), \tag{18}$$

$$\begin{aligned}[h(\lambda), f(\mu)] = & \frac{-2}{\lambda^2 - \mu^2} \left(f(\mu) - f(\lambda) \right) - \frac{2\psi\nu}{(\lambda^2 - \mu^2)\xi} \left(\mu^2 h(\mu) - \lambda^2 h(\lambda) \right) \\ & - \frac{\psi^2}{(\lambda^2 - \mu^2)\xi^2} \left(\mu^2 e(\mu) - \lambda^2 e(\lambda) \right), \end{aligned} \tag{19}$$

$$\begin{aligned}[e(\lambda), f(\mu)] = & \frac{2\psi\nu}{(\lambda^2 - \mu^2)\xi} \left(\mu^2 e(\mu) - \lambda^2 e(\lambda) \right) \\ & - \frac{4}{\lambda^2 - \mu^2} \left((\xi^2 - \mu^2\nu^2) h(\mu) - (\xi^2 - \lambda^2\nu^2) h(\lambda) \right). \end{aligned} \tag{20}$$

In the Hilbert space $\mathcal{H}$ (8), in every $V_m = \mathbb{C}^{2s+1}$ there exists a vector $\omega_m \in V_m$ such that

$$S_m^3 \omega_m = s_m \omega_m \quad \text{and} \quad S_m^+ \omega_m = 0. \tag{21}$$

We define a vector Ω_+ to be

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}. \tag{22}$$

From the definitions above, the formulas (10) - (12) and (16) it is straightforward to obtain the action of the generators $e(\lambda)$ and $h(\lambda)$ on the vector Ω_+

$$e(\lambda)\Omega_+ = 0, \quad \text{and} \quad h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+, \tag{23}$$

with

$$\rho(\lambda) = \frac{1}{\lambda} \sum_{m=1}^{N} \left(\frac{s_m}{\lambda - \alpha_m} + \frac{s_m}{\lambda + \alpha_m} \right) = \sum_{m=1}^{N} \frac{2 s_m}{\lambda^2 - \alpha_m^2}. \tag{24}$$

The generating function of the Gaudin Hamiltonians with boundary terms is given by [1]:

$$\begin{aligned}\tau(\lambda) = \operatorname{tr}_0 \mathcal{L}_0^2(\lambda) = & 2\lambda^2 \left(h^2(\lambda) + \frac{2\nu^2}{\xi^2 - \lambda^2\nu^2} h(\lambda) - \frac{h'(\lambda)}{\lambda} \right) \\ & - \frac{2\lambda^2}{\xi^2 - \lambda^2\nu^2} \left(f(\lambda) + \frac{\psi\lambda^2\nu}{\xi} h(\lambda) + \frac{\psi^2\lambda^2}{4\xi^2} e(\lambda) - \frac{\psi\nu}{\xi} \right) e(\lambda). \end{aligned} \tag{25}$$

An important initial observation in the implementation of the algebraic Bethe ansatz is that the vector Ω_+ (22) is an eigenvector of the generating function $\tau(\lambda)$. To show this we use the expression (23) and (24):

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+ = 2\lambda^2 \left(\rho^2(\lambda) + \frac{2\nu^2\, \rho(\lambda)}{\xi^2 - \lambda^2\nu^2} - \frac{\rho'(\lambda)}{\lambda} \right) \Omega_+. \tag{26}$$

With the aim of obtaining the explicit and compact form of the Bethe vectors we define the following creation operaters

$$c(\lambda) = f(\lambda) + \frac{\psi\xi}{\nu}\, h(\lambda) + \frac{\psi^2}{4\nu^2}\, e(\lambda). \tag{27}$$

Using the relations (17) - (20) it is straightforward to check that

$$[c(\lambda), c(\mu)] = 0. \tag{28}$$

Consequently, the Bethe vectors generated by the action of the operators (27) on the vector Ω_+ (22) will be symmetric functions of theirs arguments.

Our main aim is to show that the Bethe vector $\varphi_1(\mu)$ has the form

$$\varphi_1(\mu) = c(\mu)\Omega_+ = \left(f(\lambda) + \frac{\psi\xi}{\nu}\, \rho(\lambda) \right) \Omega_+, \tag{29}$$

where $c(\mu)$ is given by (27). The action of the generating function of the Gaudin Hamiltonians reads

$$\tau(\lambda)\varphi_1(\mu) = [\tau(\lambda), c(\mu)]\, \Omega_+ + \chi_0(\lambda)\varphi_1(\mu). \tag{30}$$

Using (25) and the commutation relations (17) - (20) it is evident that

$$[\tau(\lambda), c(\mu)]\, \Omega_+ = [\tau(\lambda), f(\mu)]\, \Omega_+. \tag{31}$$

Then, a straightforward calculation show that

$$\begin{aligned}[\tau(\lambda), f(\mu)]\, \Omega_+ = &-\frac{8\lambda^2}{\lambda^2-\mu^2}\left(\rho(\lambda) + \frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right)\varphi_1(\mu) \\ &+ \frac{8\lambda^2(\xi^2-\mu^2\nu^2)}{(\lambda^2-\mu^2)(\xi^2-\lambda^2\nu^2)}\left(\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2}\right)\varphi_1(\lambda).\end{aligned} \tag{32}$$

Therefore the action of the generating function $\tau(\lambda)$ on $\varphi_1(\mu)$ is given by

$$\tau(\lambda)\varphi_1(\mu) = \chi_1(\lambda,\mu)\varphi_1(\mu) + \frac{8\lambda^2(\xi^2-\mu^2\nu^2)}{(\lambda^2-\mu^2)(\xi^2-\lambda^2\nu^2)}\left(\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2}\right)\varphi_1(\lambda), \tag{33}$$

with

$$\chi_1(\lambda,\mu) = \chi_0(\lambda) - \frac{8\lambda^2}{\lambda^2-\mu^2}\left(\rho(\lambda) + \frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right). \tag{34}$$

The unwanted term in (33) vanishes when the following Bethe equation is imposed on the parameter μ,

$$\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2} = 0. \tag{35}$$

Thus we have shown that $\varphi_1(\mu)$ (29) is the desired Bethe vector of the generating function $\tau(\lambda)$ corresponding to the eigenvalue $\chi_1(\lambda,\mu)$.

3. Conclusion

We have proposed a new creation operators relevant for implementation of the algebraic Bethe ansatz for the $s\ell(2)$ Gaudin model with the general reflection matrix. However, explicit and compact form of the relevant Bethe vector $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, requires further studies and will be reported elsewhere. Such a formula would be crucial for the off shell scalar product of the Bethe vectors and these results could lead to the correlations functions of Gaudin model with boundary. Moreover, it would be of considerable interest to establish a relation between Bethe vectors and solutions of the corresponding Knizhnik-Zamolodchikov equations.

A Basic definitions

We consider the spin operators S^α with $\alpha = +, -, 3$, acting in some (spin s) representation space $\mathbb{C}^{2s+1}$ with the commutation relations

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^3, \tag{36}$$

and Casimir operator

$$c_2 = (S^3)^2 + \frac{1}{2}(S^+S^- + S^-S^+) = (S^3)^2 + S^3 + S^-S^+ = \vec{S} \cdot \vec{S}.$$

In the particular case of spin $\frac{1}{2}$ representation, one recovers the Pauli matrices

$$S^\alpha = \frac{1}{2}\sigma^\alpha = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix}.$$

We consider a spin chain with N sites with spin s representations, i.e. a local $\mathbb{C}^{2s+1}$ space at each site and the operators

$$S_m^\alpha = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1}, \tag{37}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$.

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