Killing forms on homogeneous Sasaki-Einstein manifold $T^{1,1}$ *

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Abstract

We describe the construction of Killing forms on Sasaki-Einstein spaces. Some of Killing forms are related to the contact 1-form of the Sasaki manifolds. Two additional Killing forms are associated with the complex holomorphic volume form of the Calabi-Yau metric cones of the Sasaki-Einstein manifolds. We exemplify the general scheme in the case of the five dimensional homogeneous Sasaki-Einstein space $T^{1,1}$.

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1. Introduction

In the last time Sasaki-Einstein geometry has become of significant interest in some modern developments in mathematics and theoretical physics [1, 2]. To understand the importance of Sasakian geometry we recall the fact that it sits naturally in between two Kähler geometries. On the one hand metric cones of the Sasakian manifolds are Kähler. Moreover, in the case of the Sasaki-Einstein spaces the metric cones are Ricci flat, i.e. Calabi-Yau manifolds. On the other hand any Sasakian manifold is contact and the one-dimensional foliation associated to the characteristic Reeb vector field is transversally Kähler [3].

The aim of this paper is the investigation of the hidden symmetries on toric Sasaki-Einstein manifolds. For this purpose we construct the (conformal) Killing forms using the interplay between complex coordinates of the Calabi-Yau cone and the special Killing forms on the toric Sasaki-Einstein space. We exemplify the general scheme in the case of the five dimensional $T^{1,1}$ space. The $T^{1,1}$ manifold is the first example of toric

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Sasaki-Einstein/quiver duality [4]. The explicit homogeneous $T^{1,1}$ metric was constructed in [5].

The paper is organized as follows. In the next Section we describe the relation between the Sasaki-Einstein manifolds and their Calabi-Yau cones. In Section 3 we give some definitions regarding the (conformal) Killing forms and the particular class of special Killing forms. We present the complete list of special Killing forms on Sasaki-Einstein manifolds. In Section 4 we exemplify the construction of the special Killing forms on the five-dimensional $T^{1,1}$ space. The paper ends with conclusions in Section 5.

2. Sasaki-Einstein geometry

A $(2n-1)$-dimensional manifold M is a *contact manifold* if there exists a 1-form η (called a *contact 1-form*) on M such that

$$\eta \wedge (d\eta)^{n-1} \neq 0\,. \tag{1}$$

The *Reeb vector field* ξ dual to η ($\eta = \xi^*$) satisfies

$$\eta(\xi) = 1 \quad \text{and} \quad \xi \lrcorner d\eta = 0\,. \tag{2}$$

A contact Riemannian manifold (M, g_M) is Sasakian if its metric cone $C(M)$

$$C(M) \cong \mathbb{R}_+ \times M\,,\; g_{C(M)} = dr^2 + r^2\, g_M\,, \tag{3}$$

is Kähler [1] with Kähler form [6, 7]

$$\omega = \frac{1}{2} d(r^2 \eta) = r dr \wedge \eta + \frac{1}{2} r^2 d\eta\,. \tag{4}$$

Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line $\mathbb{R}_+$. The Sasakian manifold (M, g_M) is naturally isometrically embedded into the metric cone via the inclusion

$$M = \{r = 1\} = \{1\} \times M \subset C(M)\,. \tag{5}$$

In the case of Sasaki-Einstein manifolds there is a well known fact that the following statements are equivalent [8]:

(i) (M, g_M) is Sasaki-Einstein with $\mathrm{Ric}\, g_M = 2(n-1) g_M$;

(ii) The Kähler cone $(C(M), g_{C(M)})$ is Ricci-flat ($\mathrm{Ric}\, g_{C(M)} = 0$), i.e. Calabi-Yau manifold;

(iii) The transverse Kähler structure to the Reeb foliation $\mathcal{F}_\xi$ is Kähler-Einstein with $\mathrm{Ric}\, g^T = 2n g^T$.

There are many mathematical and physical reasons to consider toric Sasaki-Einstein manifolds. For example, for $n = 3$ a toric Sasaki-Einstein manifold is dual to a *toric quiver gauge theory*.

Referring to metric cone, $C(M)$ is toric if the standard n-torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ acts effectively on it, preserving the Kähler form ω.

3. Killing forms

The *conformal Killing-Yano tensors* provide a natural generalization of Killing vector fields. Conformal Killing-Yano tensors are sometimes referred as *twistor forms* or *conformal Killing forms.*

A conformal Killing-Yano tensor of rank p on a n-dimensional Riemannian manifold (M, g_M), endowed with the Levi-Civita connection ∇, is a p-form Ψ which satisfies

$$\nabla_X \Psi - \frac{1}{p+1} X \lrcorner d\Psi + \frac{1}{n-p+1} X^* \wedge d^* \Psi = 0\,, \tag{6}$$

for any vector field X on M.

Here d and d^* are the de Rham differential and co-differential operators with respect to the Riemannian structure g_M, $\lrcorner$ is the operator dual to the wedge product $\wedge$ and X^* is the 1-form dual to the vector field X.

In component notation, the conformal Killing-Yano tensor equation is given by

$$\begin{aligned}\nabla_{(i_1}\psi_{i_2)i_3\cdots i_{p+1}} = \frac{1}{n-p+1}\Big(& g_{i_1 i_2}\nabla_j \psi^j{}_{i_3\cdots i_{p+1}} \\ & -(p-1) g_{[i_3(i_1}\nabla_j \psi^j{}_{i_2)i_4\cdots i_{p+1}]}\Big)\,.\end{aligned} \tag{7}$$

We used round and square brackets, respectively, to denote symmetrization or antisymmetrization over the indices within.

Coclosed conformal Killing forms are called *Killing forms.* If $p = 1$, the Killing forms are just dual to Killing vector fields.

A particular class of Killing forms is represented by the *special Killing forms*, which satisfy, for some constant c, the equation [9]

$$\nabla_X (d\Psi) = cX^* \wedge \Psi\,, \tag{8}$$

for any vector field X on M. It is worth mentioning the fact that the most known Killing forms are actually special.

Having in mind our interest to describe the Killing forms on Sasaki-Einstein spaces, we note the existence of a correspondence between special Killing forms defined on the Sasaki-Einstein manifold M and the parallel forms defined on its metric cone $C(M)$. More exactly, a p-dimensional differential form Ψ is a special form on M if and only if the corresponding form

$$\Psi_{cone} := r^p dr \wedge \Psi + \frac{r^{p+1}}{p+1} d\Psi\,, \tag{9}$$

is parallel on $C(M)$.

On a $(2n-1)$-dimensional Sasaki manifold with the contact 1-form η there are the following special Killing forms:

$$\Psi_k = \eta \wedge (d\eta)^k \quad , \quad k = 0, 1, \cdots, n-1\,. \tag{10}$$

Indeed they are special Killing forms (8) as Ψ_k satisfies for any vector field X and any k the additional equation [9]

$$\nabla_X (d\Psi_k) = -2(k+1)X^* \wedge \Psi_k \,. \tag{11}$$

Besides these Killing forms, there are $n-1$ closed conformal Killing forms (also called $*$-Killing forms)

$$\Phi_k = (d\eta)^k \quad , \quad k = 1, \cdots, n-1 \,. \tag{12}$$

Let us note that in the case of the Calabi-Yau cone the holonomy is $SU(n)$ and there are *two additional* parallel forms of degree n. In order to write explicitly the additional Killing forms which correspond to these parallel forms, we shall express the volume form of the metric cone in terms of the Kähler form (4)

$$d\mathcal{V} = \frac{1}{n!}\omega^n \,. \tag{13}$$

Here ω^n is the wedge product of ω with itself n times. The volume of a Kähler manifold can be also written as [9, 10]

$$d\mathcal{V} = \frac{i^n}{2^n}(-1)^{n(n-1)/2}\Omega \wedge \overline{\Omega} \,, \tag{14}$$

where Ω is the complex volume holomorphic $(n,0)$ form of $C(M)$. The additional (real) parallel forms are given by the real and respectively imaginary part of the complex volume form.

4. Special Killing forms on $C(T^{1,1})$ space

There are known various compactifications of chiral $\mathcal{N}=2$ ten-dimensional supergravity to five dimensions. The simplest compactification, apart from S^5, involve the coset spaces $T^{p,q}$ [11]. Of these, only $T^{1,1}$ preserves some supersymmetry, namely this compactification should be dual to an $\mathcal{N}=1$ superconformal field theory in four dimensions. $T^{1,1}$ is a homogeneous space $T^{1,1} = (SU(2) \times SU(2)/U(1)$ with the $U(1)$ being a diagonal subgroup of the maximal torus of $SU(2) \times SU(2)$.

The Calabi-Yau cone $C(T^{1,1})$ over the homogeneous Sasaki-Einstein manifold $T^{1,1}$ can be described by the following equation in four complex variables $z_i \in \mathbb{C}$

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \,, \tag{15}$$

with a double point singularity at $z_i = 0$. The manifold $T^{1,1}$ can be identified as the intersection of the conifold (15) and the sphere

$$\sum_{i=1}^{4} |z_i|^2 = 1 \,. \tag{16}$$

The holomorphic three form of the Calabi-Yau cone $C(T^{1,1})$ is [4]:

$$\Omega = \frac{dz_2 \wedge dz_3 \wedge dz_4}{z_1}\,, \tag{17}$$

and the metric on the conifold may be written as

$$ds^2_{C(T^{1,1})} = dr^2 + r^2 ds^2_{T^{1,1}}\,. \tag{18}$$

$T^{1,1}$ is a S^1 bundle over $S^2 \times S^2$ and the metric may be written as [5, 6]

$$\begin{aligned} ds^2(T^{1,1}) =& \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ &+ \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2\,. \end{aligned} \tag{19}$$

The coordinates (θ_1, ϕ_1) and (θ_2, ϕ_2) parametrize the sphere S^2 in a standard way and the angle $\psi \in [0, 4\pi)$ parametrizes the $U(1)$ fiber.

We have three commuting $U(1)$ actions preserving the Kähler form of $C(T^{1,1})$ generated by $\partial/\partial\phi_i$, $i = 1, 2$ and $\partial/\partial\psi$. In particular $3\partial/\partial\psi$ is the Reeb vector so that $T^{1,1}$ is a regular Sasaki-Einstein manifold and its base Kähler-Einstein manifold is just $C\mathbb{P}^1 \times C\mathbb{P}^1$ [6].

The symplectic form on the cone is

$$\begin{aligned} \omega =& -\frac{r^2}{6}(\sin\theta_1 d\theta_1 \wedge d\phi_1 + \sin\theta_2 d\theta_2 \wedge d\phi_2) \\ &+ \frac{r}{3} dr \wedge (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)\,. \end{aligned} \tag{20}$$

Taking into account (4) we could extract the contact 1-form η of the symplectic manifold $T^{1,1}$.

$$\eta = \frac{1}{3}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)\,, \tag{21}$$

and

$$d\eta = -\frac{1}{3}(\sin\theta_1 d\theta_1 \wedge d\phi_1 + \sin\theta_2 d\theta_2 \wedge d\phi_2)\,. \tag{22}$$

Using the explicit expression of the contact 1-form η we can proceed directly to the construction of the special Killing forms Ψ_k (10) and conformal Killing forms Φ_k (12).

Two additional special Killing 2-forms are associated with the complex holomorphic volume form Ω (17) of the Calabi-Yau metric cone. For this purpose we express the complex coordinates z_i in terms of the coordinates $(r, \theta_i, \phi_i, \psi)$ as follows [12]:

$$
\begin{aligned}
z_1 =& \frac{r^{3/2} e^{i\psi/2}}{\sqrt{2}} \\
&\times \left[\cos\left(\frac{\theta_1+\theta_2}{2}\right) \cos\left(\frac{\phi_1+\phi_2}{2}\right) + i \cos\left(\frac{\theta_1-\theta_2}{2}\right) \sin\left(\frac{\phi_1+\phi_2}{2}\right) \right] , \\
z_2 =& - \frac{r^{3/2} e^{i\psi/2}}{\sqrt{2}} \\
&\times \left[\cos\left(\frac{\theta_1+\theta_2}{2}\right) \sin\left(\frac{\phi_1+\phi_2}{2}\right) - i \cos\left(\frac{\theta_1-\theta_2}{2}\right) \cos\left(\frac{\phi_1+\phi_2}{2}\right) \right] , \\
z_3 =& - \frac{r^{3/2} e^{i\psi/2}}{\sqrt{2}} \\
&\times \left[\sin\left(\frac{\theta_1+\theta_2}{2}\right) \cos\left(\frac{\phi_1-\phi_2}{2}\right) - i \sin\left(\frac{\theta_1-\theta_2}{2}\right) \sin\left(\frac{\phi_1-\phi_2}{2}\right) \right] , \\
z_4 =& - \frac{r^{3/2} e^{i\psi/2}}{\sqrt{2}} \\
&\times \left[\sin\left(\frac{\theta_1+\theta_2}{2}\right) \sin\left(\frac{\phi_1-\phi_2}{2}\right) + i \sin\left(\frac{\theta_1-\theta_2}{2}\right) \cos\left(\frac{\phi_1-\phi_2}{2}\right) \right] .
\end{aligned}
\tag{23}
$$

For later use it is convenient to introduce the coordinates $\theta_\pm$ and $\phi_\pm$ instead of θ_i and ϕ_i:

$$
\theta_\pm = \frac{\theta_1 \pm \theta_2}{2} \quad , \quad \phi_\pm = \frac{\phi_1 \pm \phi_2}{2} \, . \tag{24}
$$

In terms of these coordinates, the holomorphic form (17) becomes:

$$
\begin{aligned}
\Omega =& \frac{r^3 e^{i\psi}}{2} \frac{[cos\theta_+ \cos\phi_+ - i \cos\theta_- \sin\phi_+]}{[\cos^2\theta_+ \cos^2\phi_+ + \cos^2\theta_- \sin^2\phi_+]} \\
&\times \Bigg[\frac{3}{2} \frac{dr}{r} (-\cos\theta_+ \sin\phi_+ + i \cos\theta_- \cos\phi_+) \\
&+ \frac{i}{2} d\psi (-\cos\theta_+ \sin\phi_+ + i \cos\theta_- \cos\phi_+) \\
&+ (\sin\theta_+ \sin\phi_+ d\theta_+ - \cos\theta_+ \cos\phi_+ d\phi_+) \\
&+ i(-\sin\theta_- \cos\phi_+ d\theta_- - \cos\theta_- \sin\phi_+ d\phi_+) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\wedge \left[\frac{3}{2}\frac{dr}{r}(-\sin\theta_+\cos\phi_- + i\sin\theta_-\sin\phi_-)\right.\\
&+\frac{i}{2}d\psi(-\sin\theta_+\cos\phi_- + i\sin\theta_-\sin\phi_-)\\
&+(-\cos\theta_+\cos\phi_- d\theta_+ + \sin\theta_+\sin\phi_- d\phi_-)\\
&\left.+i(\cos\theta_-\sin\phi_- d\theta_- + \sin\theta_-\cos\phi_- d\phi_-)\right]\\
&\wedge \left[\frac{3}{2}\frac{dr}{r}(-\sin\theta_+\sin\phi_- - i\sin\theta_-\cos\phi_-)\right.\\
&+\frac{i}{2}d\psi(-\sin\theta_+\sin\phi_- - i\sin\theta_-\cos\phi_-)\\
&+(-\cos\theta_+\sin\phi_- d\theta_+ - \sin\theta_+\cos\phi_- d\phi_-)\\
&\left.+i(-\cos\theta_-\cos\phi_- d\theta_- + \sin\theta_-\sin\phi_- d\phi_-)\right].
\end{aligned}
\tag{25}
$$

In accordance with (9) to this 3-form Ω on the Calabi-Yau metric cone it corresponds a 2-form Ψ on the Sasaki-Einstein space $T^{1,1}$:

$$
\Omega = r^2 dr \wedge \Psi + \frac{r^3}{3} d\Psi\,. \tag{26}
$$

After a straightforward calculation we obtain:

$$
\begin{aligned}
\Psi =& \frac{3}{4}e^{i\psi}\frac{[cos\theta_+\cos\phi_+ - i\cos\theta_-\sin\phi_+]}{[\cos^2\theta_+\cos^2\phi_+ + \cos^2\theta_-\sin^2\phi_+]}\\
&\times[-(\cos\theta_+\cos\phi_+ + i\cos\theta_-\sin\phi_+)d\theta_+\wedge d\theta_-\\
&\quad + \cos\theta_+\sin\theta_-(-\cos\theta_-\sin\phi_+ + i\cos\theta_+\cos\phi_+)d\theta_+\wedge d\phi_+\\
&\quad + \sin\theta_+\cos\theta_-(-\cos\theta_-\sin\phi_+ + i\cos\theta_+\cos\phi_+)d\theta_+\wedge d\phi_-\\
&\quad + \sin\theta_+\cos\theta_-(\cos\theta_-\sin\phi_+ - i\cos\theta_+\cos\phi_+)d\theta_-\wedge d\phi_+\\
&\quad + \cos\theta_+\sin\theta_-(\cos\theta_-\sin\phi_+ - i\cos\theta_+\cos\phi_+)d\theta_-\wedge d\phi_-\\
&\quad + (\sin^2\theta_+ - \sin^2\theta_-)(\cos\theta_+\cos\phi_+ + i\cos\theta_-\sin\phi_+)d\phi_+\wedge d\phi_-]\,.
\end{aligned}
\tag{27}
$$

From here, calculating the real and imaginary part of Ψ we get two new special Killing 2-forms.

5. Conclusions

The importance of Killing forms comes from their role in physics being related to hidden symmetries, superintegrability of field equations, quantum symmetry operators, supersymmetries, etc.

In general it is a hard task to solve the generalized Killing equation (6). In addition to the brute force method of trying to solve (6) in some cases there are other methods which are considerable simpler. In the case of Sasaki-Einstein spaces the complete set of special Killing forms is related to the geometrical structures. Special Killing forms are constructed from the contact 1-form of the Sasaki-Einstein spaces and other additional Killing forms are associated with the complex volume form of the Calabi-Yau metric cone.

In this paper we give a method to construct special Killing forms on a toric Sasaki geometry and exemplify the general scheme in the case of the five dimensional homogeneous Sasaki-Einstein manifold $T^{1,1}$. Many examples of Sasaki-Einstein manifolds may be obtain via toric geometry [7, 13, 14, 15], and such examples are a good testing ground for the AdS/CFT correspondence [16].

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