Lie symmetries of the Wheeler-DeWitt equation with application in Hybrid Gravity

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Abstract

Lie point symmetries for Hybrid Gravity models are considered in order to select potentials and to find invariant solutions of the Wheeler-DeWitt equation. Analytical solution of the modified field equations determines the Hubble function H^2 as a fourth order polynomial with five non-vanishing coefficients. It can be reinterpreted in terms of well-known, from the extended Standard Cosmological Model (ΛCDM), perfect fluid components.

1. Introduction

Although General Relativity (GR) is a beautiful theory verified by numerous astronomical observations, there are still phenomena such as late-time cosmic acceleration [1, 2, 3], dark matter and dark energy [4] that the standard theory is not able to explain. It seems that some modifications of GR are necessary to answer that questions. One of them is $f(R)$ gravity [5, 6] which instead of the Ricci scalar R in the Hilbert-Einstein Lagrangian takes an unknown function of R. Such a consideration can be treated in two formalisms: metric and Palatini one (see e.g. [7] and references there in) where in the former the metric and connection are considered as two independent variables and curvature $\mathcal{R}$ is constructed of the

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connection. Both formalisms can be combined in the approach denoted as hybrid metric-Palatini gravity or $f(X)$ gravity [8, 9], where the action is taken as the standard Hilbert-Einstein case (linear in Ricci scalar R) with a nonlinear term in the Palatini curvature scalar $\mathcal{R}$. The equations of motions and Wheeler-DeWitt (WDW) equation of that theory can be examined by using point symmetries which help to solve the given differential equations. It is also possible to find potentials appearing in the model which are distinguished by Noether symmetries. Here, we will present the Hybrid Gravity model in spatially flat Friedmann-Robertson-Walker spacetime with solutions of the classical field equations and invariant solutions of WDW equation.

2. Symmetries of differential equations

A Lie symmetry of a differential equation is a one parameter point transformation which leaves the differential equation invariant. Lie symmetries is the main tool to study nonlinear differential equations. Indeed, Lie symmetries provide invariant functions which can be used to construct analytic solutions for a differential equation. For differential equations which arise from a variational principle there may exist a special class of Lie symmetries which are called Noether symmetries. Noether symmetries are the generators of the one parameter point transformation which transform the Lagrangian such as the Euler-Lagrange equation to be invariant. The importance of Noether symmetries is that they provide use of first integrals/conservation laws [10].

Consider the second order partial differential equation (PDE) $\mathcal{H} \equiv \mathcal{H}(x^i, u^A, u^B_{,i}, u^C_{,ij}) = 0$ where x^i are the independent variables and u^A are the dependent ones. We shall say that the PDE $\mathcal{H}$ is invariant under the action of the infinitesimal point transformation

$$\bar{x}^i = x^i + \varepsilon \xi^i(x^k, u^B) \ , \ \bar{u}^A = u^A + \varepsilon \eta^A(x^k, u^B) \tag{1}$$

with infinitesimal generator

$$\mathbf{X} = \xi^i(x^k, u^B)\partial_{x^i} + \eta^A(x^k, u^B)\partial_{u^A} \tag{2}$$

if and only if there exists a function λ such that the following condition holds [11]

$$\mathbf{X}^{[2]}(\mathcal{H}) = \lambda\mathcal{H} \ , \ mod\mathcal{H} = 0, \tag{3}$$

where $\mathbf{X}^{[2]} = \mathbf{X} + \eta^A_i \partial_{u^A_i} + \eta^A_{ij}\partial_{u^A_{ij}}$ is the second prolongation vector of $\mathbf{X}$ and[1]

$$\eta^A_i = D_i\eta^A - u^A_j D_i\xi^j, \ \ \eta^A_{ij} = D_i\eta^A_j - u^A_{ik} D_j \xi^k. \tag{4}$$

For PDEs arising from a variational principle, Noether's theorem states [12] that the action of the generator (2) of the infinitesimal transformation (1) acting on the Lagrangian $L = L(x^k, u, u_k)$ leaves a field equation

[1]Where $D_i = \partial_i + u_i\partial_u + u_{ij}\partial_{u_j} + ...$ is the total derivative.

$\mathcal{H}(x^i, u, u_{,i}, u_{,ij}) = 0$ invariant if there exists a vector field $A^i = A^i(x^i, u)$ such that the following condition is satisfied ($\mathbf{X}^{[1]} = \mathbf{X} + \eta_i \partial_{u_i}$):

$$\mathbf{X}^{[1]} L + L D_i \xi^i = D_i A^i \ . \tag{5}$$

The corresponding Noether flow I^i is defined by the expression

$$I^i = \xi^k \left(u_k \frac{\partial L}{\partial u_i} - L \right) - \eta \frac{\partial L}{\partial u_i} + A^i \tag{6}$$

being conserved if the relation $D_i I^i = 0$ is satisfied.

The method of the Noether symmetries of the field has been applied by many authors as a selection rule of the unknown functions in various cosmological models, e.g. in scalar field cosmology, in f(R) gravity and others (for instance see [13, 14, 15, 16, 17, 18, 19]). However in [20, 21] it has been shown that this selection rule has geometrical properties since the Noether symmetries are related with the collineations of the minisuperspace. Recently, in [22] it has been proposed that the cosmological model will be determined by the existence of Lie symmetries of the Wheeler-DeWitt equations (WDW) in quantum cosmology. This selection rule is more general and it is related with the Noether symmetry criterion.

In the following we will apply this method in order to determine exact solutions of the WDW equation by using the Lie symmetries in Hybrid Gravity. Furthermore, we will show that when the WDW equation is separable the classical field equations are integrable and vice versa.

3. Hybrid Gravity

The action of the hybrid metric-Palatini gravity has the following form [8, 9]:

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4 x \sqrt{-g} [R + f(\mathcal{R})] + S_m \tag{7}$$

where R is the metric curvature scalar and $f(\mathcal{R})$ is a function of the Palatini curvature scalar which is constructed by an independent torsionless connection $\hat{\Gamma}$. One uses the metric g in order to construct the scalar from the Ricci tensor of $\hat{\Gamma}$, i.e. $\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu}(\hat{\Gamma})$. The variation of the above action with respect to the connection gives modified gravitational field equations

$$G_{\mu\nu} + f'(\mathcal{R}) \mathcal{R}_{\mu\nu} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{8}$$

where $G_{\mu\nu}$ is the Einstein tensor for the metric $g_{\mu\nu}$ (with Lorentzian signature). It can be shown that $R_{\mu\nu}(g)$ and $\mathcal{R}_{\mu\nu}(h)$ are conformally related to each other by the relation $h_{\mu\nu} = f'(\mathcal{R}) g_{\mu\nu}$, where the conformal factor is given by $f'(\mathcal{R}) = df(\mathcal{R})/d\mathcal{R}$. The trace of Eq. (8) is the hybrid structural equation, where one can algebraically express the Palatini curvature $\mathcal{R}$ in terms of a quantity X assuming that the following equation has analytic solutions:

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa^2 T + R \equiv X. \tag{9}$$

The variable X measures the deviation from the General Relativity trace equation $R = -\kappa^2 T$, that is GR is fully recovered for $X = 0$ [9].

As for the pure metric and Palatini case [5, 7], the above action can be transformed into a scalar-tensor theory if $f''(\mathcal{R}) \neq 0$

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}[R + \phi\mathcal{R} - V(\phi)]. \tag{10}$$

where $\phi \equiv f'(\mathcal{R})$ and $V(\phi) = \mathcal{R}f'(\mathcal{R}) - f(\mathcal{R})$. Using the relation between R and $\mathcal{R}$, given by $\mathcal{R} = R + \frac{3}{2\phi^2}\partial_\mu\phi\partial^\mu\phi - \frac{3}{\phi}\Box\phi$ (see [9] for details) one finally obtains the standard scalar-tensor form

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[(1+\phi)R + \frac{3}{2\phi}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right]. \tag{11}$$

Let us consider the FRW spatially flat metric

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right). \tag{12}$$

Therefore, from action (11) one deduces the pointlike Lagrangian

$$\mathcal{L} = \frac{1}{N}\left(6a\dot{a}^2(1+\phi) + 6a^2\dot{a}\dot{\phi} + \frac{3}{2\phi}a^3\dot{\phi}^2\right) + Na^3V(\phi). \tag{13}$$

The energy condition resulting from the above Lagrangian $\left(\frac{\partial L}{\partial N} = 0\right)$ is

$$\frac{1}{N^2}\left(6a\dot{a}^2(1+\phi) + 6a^2\dot{a}\dot{\phi} + \frac{3}{2\phi}a^3\dot{\phi}^2\right) - a^3V(\phi) = 0. \tag{14}$$

The details concerning the Friedmann and field equations of (13) can be find in [23].

From Lagrangian (13) we define the momentum p_a, p_ϕ as follows

$$p_a = \frac{6a}{N}\left(2\left(1+\phi\right)\dot{a} + a\dot{\phi}\right)\ ,\ p_\phi = \frac{3a^2}{\phi N}\left(2\phi\dot{a} + a\dot{\phi}\right) \tag{15}$$

hence the energy condition (14) becomes

$$\frac{1}{12a}p_a^2 - \frac{\phi}{3a^2}p_a p_\phi + \frac{\phi\left(1+\phi\right)}{3a^3}p_\phi^2 - a^3V\left(\phi\right) = 0. \tag{16}$$

and the rest of the field equations are the Hamilton's equations of (16).

4. Symmetries of the WDW equation

Roughly speaking, WDW equation is a quantized version of a Hamiltonian of a considered system. In our case, since the minisuperspace is two-dimensional described by the minisuperspace metric G_{ij}, the WDW it has the following form

$$\Delta\Psi - a^3 V(\phi)\Psi = 0, \tag{17}$$

where $\Delta = \frac{1}{\sqrt{|G|}}\frac{\partial}{\partial x^i}\left(\sqrt{|G|}G^{ij}\frac{\partial}{\partial x^i}\right)$ is the Laplace operator, Ψ is a wave function of the Universe and $x^i = \{a, \phi\}$. Using Lie symmetries for the Lagrangian (13) one can find the form of the potential which in that case is a constant one [23].

Since the dynamical system (13) is conformally invariant, we can apply the conformal transformation $\mathrm{d}\tau = N(a)\mathrm{d}t$ to the system and find new solutions [24, 25, 26]. Therefore, one considers the conformal Lagrangian

$$\mathcal{L}\left(x^i, (x^i)'\right) = \frac{a^3 V(\phi)}{N(a)} + N(a)\left[6a(1+\phi)a'^2 + 6a^2 a'\phi' + \frac{3}{2\phi}a^3\phi'^2\right], \tag{18}$$

where the prime denotes $d/d\tau$. In order to simplify the problem we chose $N(a) = a^{-1}e^{N_0 a}$ for which we found two extra Noether symmetries with potentials (V_0 and V_1 are constant):

$$X_1 = -\frac{1}{2}\partial_a + \frac{\phi + V_1\sqrt{\phi}}{a}\partial_\phi, \quad V(\phi) = V_0\left(\sqrt{\phi} + V_1\right)^4; \tag{19}$$

$$X_2 = 2\tau\partial_\tau + a\left(\sqrt{\phi}V_1 + 1\right)\partial_a - 2V_1\sqrt{\phi}(\phi+1)\partial_\phi, \tag{20}$$

$$V(\phi) = V_0(1+\phi)^2\exp\left(\frac{6}{V_1}\arctan\sqrt{\phi}\right). \tag{21}$$

The corresponding conservation laws are

$$I_1 = 6\left(V_1\sqrt{\phi} - 1\right)\dot{a} + 3\frac{a}{\sqrt{\phi}}V_1\dot{\phi}, \tag{22}$$

$$I_2 = 12a(1+\phi)\dot{a} + 6a^2\left(1 - \frac{V_1}{\sqrt{\phi}}\right)\dot{\phi}. \tag{23}$$

4.1. Exact solutions of WDW equation

In this paper we are focused on the case of power law potential (19). It has been shown [23] that after a coordinate transformation the Lagrangian (18) turns out to be

$$\mathcal{L}\left(x, y, x', y'\right) = \frac{1}{2}x'^2 + \frac{1}{2}y'^2 + \bar{V}_0 y^4, \tag{24}$$

where $\bar{V}_0$ is a constant and (x, y) are appropriate functions of (a, ϕ). The Hamiltonian of the field equations is given by

$$\tilde{H} = \frac{1}{2}p_x'^2 + \frac{1}{2}p_y'^2 - \bar{V}_0 y^4, \tag{25}$$

where p_x, p_y are the momenta. The WDW equation constructed from (25) reads

$$\Psi_{,xx} + \Psi_{,yy} - 2\bar{V}_0 y^4 \Psi = 0 \tag{26}$$

and it admits Lie point symmetries for the vector fields [26]

$$X_\Psi = c_1 \partial_x + (c_2 \Psi + b(x, y)) \partial_\Psi, \quad X_b = b(x, y) \partial_\Psi, \tag{27}$$

where $b(x, y)$ is a function that satisfies (26). In order to reduce (26) and to find its solutions one may use the zeroth order invariants. The invariant solutions of the Wheeler-DeWitt equation is found to be

$$\Psi(x, y) = \sum_\mu \left[y_1 e^{\mu x + w(y)} + y_2 e^{\mu x - w(y)} \right], \tag{28}$$

where $w(y) = \frac{\sqrt{2}}{2} \int \sqrt{\left(2\bar{V}_0 y^4 - \mu^2\right)} \mathrm{d}y$ and $\mu \in \mathbb{C}$.

4.2. Classical solutions

Let us come back to the Hamiltonian (25) which has a constraint, i.e. $\tilde{H} = 0$. The Hamilton-Jacobi equation provides the action

$$S = c_1 x + \int \sqrt{2\bar{V}_0 y^4 - c_1^2} + S_0, \tag{29}$$

which allows us to reduce the field equations of (25) and find their exact solutions of the form

$$x(\tau) = x_1 \tau + x_2, \quad \int \frac{dy}{\sqrt{2\bar{V}_0 y^4 - c_1^2}} = \varepsilon (\tau - \tau_0), \tag{30}$$

where $\varepsilon = \pm 1$. Considering the case $V_1 = 0$, one has the solution $a(\tau) = a_3 \tau$ where $\tau = \sqrt{t}$ which is the radiation solution. Assuming $V_1 \neq 0$ and $c_1 = 0$, the scale factor reads instead

$$a(\tau) = a_1 - a_2 \frac{1}{\tau - \tau_0} + a_3 (\tau - \tau_0), \tag{31}$$

where a_3, a_1, a_2, are constants. One obtains the Hubble function[2]:

$$H(a) = a'^2 = a_3 a^{-2} + 4a_3^2 a_2 \left(a^3 - a_1 a^2 + \varepsilon a^2 \sqrt{(a - a_1)^2 + 4a_3 a_2} \right)^{-2}, \tag{32}$$

[2] Recall that $H = \frac{1}{a}\frac{da}{dt} = \frac{1}{a^2}\frac{da}{d\tau}$

where $(a-a_1)^2+4a_3a_2 \geq 0$ and $a(\tau) \in \mathbb{R}$ must be satisfied for a real solution. Putting $a_3=0$ one finds that $a_2^{-1}=H_0/(|a_1|+1)^2$ so the Hubble function can be written as (at the present time $a_0=1$)

$$\frac{H^2(a)}{H_0^2}=\Omega_r a^{-4}+\Omega_m a^{-3}+\Omega_k a^{-2}+\Omega_f a^{-1}+\Omega_\Lambda, \tag{33}$$

where

$$\Omega_f=\frac{|4a_1|}{(|a_1|+1)^4}, \qquad \Omega_\Lambda=\frac{1}{(|a_1|+1)^4}, \qquad \Omega_r=\frac{|a_1|^4}{(|a_1|+1)^4},$$
$$\Omega_m=\frac{|4a_1|^3}{(|a_1|+1)^4}, \quad \Omega_k=\frac{|6a_1|^2}{(|a_1|+1)^4}. \tag{34}$$

Each power of $\sqrt{\phi}$ in the potential above introduces a power of the scale factor in the Hubble function (33) of the model. It in turn allows perfect fluid interpretation as: radiation ($w=1/3$), dust ($w=0$), curvature-like fluid ($w=-1/3$), a (non-phantom) dark energy fluid ($w=-2/3$) and a cosmological constant ($w=-1$), where $w=p/\rho$ denotes the corresponding barotropic equation of state parameter. It should be noticed that we assumed spatially flat universe but the curvature-like term follows from the Hybrid Gravity.

5. Conclusions

The aim of the paper was to discuss applications of point symmetries in Hybrid Gravity model and the most important results from [23]. One needed to apply conformal transformation to the Lagrangian in order to obtain new classes of solutions which one of them was considered in the paper. For the power law potential $V(\phi)=V_0\left(\sqrt{\phi}+V_1\right)$ the Hubble function is a fourth-order polynomial with non-vanishing coefficients, i.e. each power of $\sqrt{\phi}$ produces a corresponding fluid in the model. The special form of the potential ($V_1=0$) gives radiation solutions. In that case it is possible to write WDW equation in a form allowing to find explicitly (using the zeroth order invariants) an invariant solution. This is Wave Function of the Universe [27] whose peaks appear for conserved momenta and observed universe (i.e. classical cosmological solution) [28].

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