Weakly curved background T-duals

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Outline

- Closed string in a weakly curved background
- Generalized T-dualization procedure
- T-dual backgrounds
- T-duality laws
- T-dualization diagram

Bosonic string in a weakly curved background

- Action for the closed string in the conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$

$$S[x] = \kappa \int_\Sigma d^2\xi \; \partial_+ x^\mu \Pi_{+\mu\nu}(x) \partial_- x^\nu, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma$$

- Background consists of metric tensor $G_{\mu\nu} = G_{\nu\mu}$ and Kalb-Ramond field $B_{\mu\nu} = -B_{\nu\mu}$

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x)$$

- Space-time equations of motion

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0, \quad D_\rho B^\rho{}_{\mu\nu} = 0$$

- Weakly curved background

$$G_{\mu\nu}(x) = const, \; B_{\mu\nu}(x) = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho, \quad b_{\mu\nu}, B_{\mu\nu\rho} = const$$

Generalized Buscher construction of a T-dual theory

- Old steps (applicable to backgrounds which do not depend on the coordinates which one T-dualizes):
  - Localize the global symmetry $\delta x^\mu = \lambda^\mu = const$
  - Introduce the gauge fields v_α^μ
  - Substitute the ordinary derivatives with the covariant ones

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v_\alpha^\mu$$

  - Impose the transformation law for the gauge fields

$$\delta v_\alpha^\mu = -\partial_\alpha \lambda^\mu, \quad (\lambda^\mu = \lambda^\mu(\tau, \sigma))$$

- New step (enables T-dualization of every coordinate):
  - Substitute the coordinate x^μ by the invariant coordinate

$$\Delta x_{inv}^\mu \equiv \int_P d\xi^\alpha \, D_\alpha x^\mu = x^\mu - x^\mu(\xi_0) + \Delta V^\mu,$$

here

$$\Delta V^\mu \equiv \int_P d\xi^\alpha v_\alpha^\mu.$$

Generalized Buscher construction

- Old step:
  - Require the equivalence with the initial theory
  - Field strength

    $$F^\mu_{\alpha\beta} \equiv \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha$$

    must be zero
  - Add the Lagrange multiplier y_μ term in the Lagrangian
- Result:
  - Gauge invariant action

    $$S_{inv} = \kappa \int d^2\xi \Big[D_+ x^\mu \Pi_{+\mu\nu}(\Delta x_{inv}) D_- x^\nu + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu) \Big]$$

  - Fix the gauge $x^\mu(\xi) = x^\mu(\xi_0)$
  - Gauge fixed action

    $$S_{fix}[y, v_\pm] = \kappa \int d^2\xi \Big[v^\mu_+ \Pi_{+\mu\nu}(\Delta V) v^\nu_- + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu) \Big]$$

Gauge fixed action

- Two equations of motion can direct the procedure either back to the initial action or forward to the T-dual action.
- For the equation of motion obtained varying the action over the Lagrange multipliers, the gauge fixed action reduces to the initial action.
- For the equation of motion obtained varying the action over the gauge fields one obtains the T-dual theory.
- Comparing the solutions for the gauge fields in these two directions, one obtains the T-dual coordinate transformation laws.

Complete T-dualization

- T-dual action

$$S[y] = \frac{\kappa^2}{2} \int d^2\xi \; \partial_+ y_\mu \Theta_-^{\mu\nu}\left(\Delta V^{(0)}(y)\right) \partial_- y_\nu$$

- T-dual background

$$\Theta_-^{\mu\nu} = -\frac{2}{\kappa}\left(G_E^{-1} \Pi_- G^{-1}\right)^{\mu\nu},$$

where $G^E_{\mu\nu} \equiv [G - 4BG^{-1}B]_{\mu\nu}$

- Argument $V^\mu = -\kappa\, \theta_0^{\mu\nu} y_\nu + (g^{-1})^{\mu\nu} \tilde{y}_\nu$

Original theory $S[x]$	$\longrightarrow$	*T-dual theory* $S[y]$
Noether current j^α_μ *Conservation law = Equation of motion* $\partial_\alpha j^\alpha_\mu = 0$		*Topological current* ${}^\star i^\alpha_\mu = -\kappa \epsilon^{\alpha\beta} \partial_\beta y_\mu$ *Conservation law = Bianchi identity* $\partial_\alpha {}^\star i^\alpha_\mu = 0$
T-dual of T-dual theory $S[x]$	$\longleftarrow$	*T-dual theory* $S[y]$
Topological current $i^{\alpha\mu} = -\kappa\epsilon^{\alpha\beta}\partial_\beta x^\mu$ *Conservation law = Bianchi identity* $\partial_\alpha i^{\alpha\mu} = 0$		*Noether current* ${}^\star j^{\alpha\mu}$ *Conservation law = Equation of motion* $\partial_\alpha {}^\star j^{\alpha\mu} = 0$

Partial T-dualizations

- T-dualization along direction x^μ: T^μ
- T-dualization along dual direction y_μ: T_μ
- T-dualization along initial directions

$$\mathcal{T}^a = \circ_{n=1}^{d} T^{\mu_n}, \qquad \mathcal{T}^i = \circ_{n=d+1}^{D} T^{\mu_n}, \qquad \mathcal{T} = \circ_{n=1}^{D} T^{\mu_n}$$

- T-dualization along dual directions

$$\mathcal{T}_a = \circ_{n=1}^{d} T_{\mu_n}, \qquad \mathcal{T}_i = \circ_{n=d+1}^{D} T_{\mu_n}, \qquad \widetilde{\mathcal{T}} = \circ_{n=1}^{D} T_{\mu_n}$$

$\mu_n \in (0, 1, \ldots, D-1)$

T-duality diagram

- We show

$$\mathcal{T}^i \circ \mathcal{T}^a = \mathcal{T}, \quad \mathcal{T}_i \circ \mathcal{T}_a = \widetilde{\mathcal{T}}, \quad \mathcal{T}_a \circ \mathcal{T}^a = 1$$

$$\begin{tikzcd}
& S[x^i, y_a] \arrow[dl, shift left, "\mathcal{T}_a"] \arrow[dr, shift left, "\mathcal{T}^i"] & \\
S[x^\mu] \arrow[ur, shift left, "\mathcal{T}^a"] \arrow[rr, shift left, "\mathcal{T}"] & & S[y_\mu] \arrow[ul, shift left, "\mathcal{T}_i"] \arrow[ll, shift left, "\widetilde{\mathcal{T}}"]
\end{tikzcd}$$

$\mathcal{T}^a : S[x] \to S[x^i, y_a]$

$$
\begin{aligned}
S_{fix}[x^i, v^a_\pm, y_a] \;&=\; \kappa \int d^2\xi \Big[\partial_+ x^i \Pi_{+ij}(x^i, \Delta V^a)\partial_- x^j \\
&+\; \partial_+ x^i \Pi_{+ia}(x^i, \Delta V^a) v^a_- + v^a_+ \Pi_{+ai}(x^i, \Delta V^a)\partial_- x^i \\
&+\; v^a_+ \Pi_{+ab}(x^i, \Delta V^a) v^b_- + \frac{1}{2}(v^a_+ \partial_- y_a - v^a_- \partial_+ y_a)\Big]
\end{aligned}
$$

Equations of motion:

$$\partial_+ v^a_- - \partial_- v^a_+ = 0$$

$$\Pi_{\pm ai}(x^i, \Delta V^a)\partial_\mp x^i + \Pi_{\pm ab}(x^i, \Delta V^a) v^b_\mp + \frac{1}{2}\partial_\mp y_a = \pm\beta^\pm_a(x^i, V^a)$$

T-duality laws

- $\mathcal{T}^a : S[x] \to S[x^i, y_a]$
  - $\partial_\mp x^a \cong$ $-2\kappa\tilde{\Theta}^{ab}_\mp\big(x^i, \Delta V^a(x^i, y_a)\big)\Big[\Pi_{\pm bi}\big(x^i, \Delta V^a(x^i, y_a)\big)\partial_\mp x^i + \frac{1}{2}\partial_\mp y_b \mp \beta^\pm_b\big(x^i, V^a(x^i, y_a)\big)\Big]$
  - $x^{(0)a} \cong V^{(0)a}(x^i, y_a)$
- $\mathcal{T}_a : S[x^i, y_a] \to S[x]$
  - $\partial_\mp y_a \cong -2\Pi_{\pm a\mu}(x)\partial_\mp x^\mu \pm 2\beta^\pm_a(x)$,
  - $y_a^{(0)} \cong U_a^{(0)}(x)$

Action $S[x^i, y_a]$

$$
\begin{aligned}
S[x^i, y_a] \;=\; & \kappa \int d^2\xi \Big[\partial_+ x^i \overline{\Pi}_{+ij}\big(x^i, \Delta V^a(x^i, y_a)\big) \partial_- x^j \\
& -\kappa\, \partial_+ x^i \Pi_{+ia}\big(x^i, \Delta V^a(x^i, y_a)\big) \tilde{\Theta}_-^{ab}\big(x^i, \Delta V^a(x^i, y_a)\big) \partial_- y_b \\
& +\kappa\, \partial_+ y_a \tilde{\Theta}_-^{ab}\big(x^i, \Delta V^a(x^i, y_a)\big) \Pi_{+bi}\big(x^i, \Delta V^a(x^i, y_a)\big) \partial_- x^i \\
& +\frac{\kappa}{2}\, \partial_+ y_a \tilde{\Theta}_-^{ab}\big(x^i, \Delta V^a(x^i, y_a)\big) \partial_- y_b \Big]
\end{aligned}
$$

Argument:

$$
\begin{aligned}
\Delta V^{(0)a}(x^i, y_a) \;=\; & -\kappa \Big[\tilde{\Theta}_{0+}^{ab} \Pi_{0-bi} + \tilde{\Theta}_{0-}^{ab} \Pi_{0+bi} \Big] \Delta x^{(0)i} \\
& - \kappa \Big[\tilde{\Theta}_{0+}^{ab} \Pi_{0-bi} - \tilde{\Theta}_{0-}^{ab} \Pi_{0+bi} \Big] \Delta \tilde{x}^{(0)i} \\
& - \frac{\kappa}{2} \Big[\tilde{\Theta}_{0+}^{ab} + \tilde{\Theta}_{0-}^{ab} \Big] \Delta y_b^{(0)} - \frac{\kappa}{2} \Big[\tilde{\Theta}_{0+}^{ab} - \tilde{\Theta}_{0-}^{ab} \Big] \Delta \tilde{y}_b^{(0)}
\end{aligned}
$$

T-dual background fields

- Inverses:
  - $\tilde{\Theta}^{ab}_{\pm}\Pi_{\mp bc} = \Pi_{\mp cb}\tilde{\Theta}^{ba}_{\pm} = \frac{1}{2\kappa}\delta^a_c$
  - $\overline{\Pi}_{\pm ij}\Theta^{jk}_{\mp} = \Theta^{kj}_{\mp}\overline{\Pi}_{\pm ji} = \frac{1}{2\kappa}\delta^k_i$
- Effective metric
  $\tilde{G}_{Eab} \equiv G_{ab} - 4B_{ac}(\tilde{G}^{-1})^{cd}B_{db}$
- Noncommutativity parameter
  $\tilde{\theta}^{ab} \equiv -\frac{2}{\kappa}(\tilde{G}_E^{-1})^{ac}B_{cd}(\tilde{G}^{-1})^{db}$
- $\tilde{\Theta}^{ab}_{\pm} = \tilde{\theta}^{ab} \mp \frac{1}{\kappa}(\tilde{G}_E^{-1})^{ab}$
- $\overline{\Pi}_{+ij} \equiv \Pi_{+ij} - 2\kappa\Pi_{+ia}\tilde{\Theta}^{ab}_{-}\Pi_{+bj}$

T-dual background

- $G_{ij} = \overline{G}_{ij} = G_{ij} - G_{ia}(\tilde{G}_E^{-1})^{ab} G_{bj}$
  $-2\kappa\Big(B_{ia}\tilde{\theta}^{ab}G_{bj} + G_{ia}\tilde{\theta}^{ab}B_{bj}\Big) - 4B_{ia}(\tilde{G}_E^{-1})^{ab}B_{bj}$
- $B_{ij} = \overline{B}_{ij} = B_{ij} - \frac{\kappa}{2}G_{ia}\tilde{\theta}^{ab}G_{bj} - B_{ia}(\tilde{G}_E^{-1})^{ab}G_{bj}$
  $-G_{ia}(\tilde{G}_E^{-1})^{ab}B_{bj} - 2\kappa B_{ia}\tilde{\theta}^{ab}B_{bj}$
- $G^{ab} = (\tilde{G}_E^{-1})^{ab}$
- $B^{ab} = \frac{\kappa}{2}\tilde{\theta}^{ab}$
- $G^a{}_i = \kappa\tilde{\theta}^{ab}G_{bi} + 2(\tilde{G}_E^{-1})^{ab}B_{bi}$
- $B^a{}_i = \kappa\tilde{\theta}^{ab}B_{bi} + \frac{1}{2}(\tilde{G}_E^{-1})^{ab}G_{bi}$

T-dual backgrounds

- geometric background
  $\Pi_{+\mu\nu}$
- nongeometric background
  •$\Pi_{+ij} = \overline{\Pi}_{+ij} = \Pi_{+ij} - 2\kappa \Pi_{+ia} \tilde{\Theta}_{-}^{ab} \Pi_{+bj}$
  •$\Pi_{+i}{}^{a} = -\kappa \Pi_{+ib} \tilde{\Theta}_{-}^{ba}$
  •$\Pi^{a}_{+i} = \kappa \tilde{\Theta}_{-}^{ab} \Pi_{+bi}$
  •$\Pi_{+}^{ab} = \frac{\kappa}{2} \tilde{\Theta}_{-}^{ab}$
- nongeometric background
  $\Theta_{-}^{\mu\nu}$

Further investigations

- Poisson structures
  - definition
  - connection
- Non-commutativity relations
- Non-commutativity parameters