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Intertwining Operator Realization of anti de Sitter Holography

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Introduction

For the last fifteen years due to remarkable proposal of Maldacena the AdS/CFT correspondence is a dominant subject in string theory and conformal field theory. That proposal of was further elaborated by Polyakov et al and by Witten. After these initial papers there was an explosion of related research which continues also currently.

Let us remind that the AdS/CFT correspondence has 2 ingredients:
1. the holography principle, which is very old, and means the reconstruction of some objects in the bulk (that may be classical or quantum) from some objects on the boundary;
2. the reconstruction of quantum objects, like 2-point functions on the boundary, from appropriate actions on the bulk.

Our focus is on the first ingredient and we consider explicitly the simplest case of the (3+1)-dimensional bulk with boundary 3D Minkowski space-time. The reason for this is that until now the explicit presentation of the holography principle was realized in the Euclidean case, relying on Wick rotations of the final results.

Yet it is desirable to show the holography principle by direct construction in Minkowski space-time. This is what we do in the present paper using representation theory only. For this we use a method that is used in the mathematical literature for the construction of discrete series representations of real semisimple Lie groups [Hotta,Schmid], and which method was applied in the physics literature first in [DMPPT] exactly in the Euclidean AdS/CFT setting, though that term was not used then. This method was applied recently also to the case of non-relativistic holography [Ai-Do].

The method utilizes the fact that in the bulk the Casimir operators are not fixed numerically. Thus, when a vector-field realization of the anti de Sitter algebra so(3,2) is substituted in the bulk Casimir it turns into a differential operator. In contrast, the boundary Casimir operators are fixed by the quantum numbers of the fields under consideration. Then the bulk/boundary correspondence forces an eigenvalue equation involving the Casimir differential operator. That eigenvalue equation is used to find the two-point Green function in the bulk which is then used to construct the boundary-to-bulk integral operator. This operator maps a boundary field to a bulk field. This is our main result.

What is also important in our approach is that we show that this operator is an intertwining operator, namely, it intertwines the two representations of the anti de Sitter algebra so(3,2) acting in the bulk and on the boundary.

This also helps us to establish that each bulk field has actually two bulk-to-boundary limits. The two boundary fields have conjugated conformal weights Δ, $3 - \Delta$, and they are related by a boundary two-point function.

Preliminaries

Lie algebra and group

We need some well-known preliminaries to set up our notation and conventions. The Lie algebra $\mathcal{G}$ = so(3,2) may be defined as the set of 5×5 matrices X which fulfil the relation:*

$$ {}^{t}X\eta + \eta X = 0, \tag{1} $$

where η is given by

$$ \eta = (\eta_{AB}) = \mathrm{diag}(-1,1,1,1,-1), \qquad A,B = 0,1,\cdots,4 \tag{2} $$

Then we can choose a basis $X_{AB} = -X_{BA}$ of $\mathcal{G}$ satisfying the commutation relations

$$ [X_{AB}, X_{CD}] = \eta_{AC}X_{BD} + \eta_{BD}X_{AC} - \eta_{AD}X_{BC} - \eta_{BC}X_{AD} \tag{3} $$

*For other purposes it may be more convenient to use the other fundamental representation in terms of 4×4 matrices.

We list the important subalgebras of $\mathcal{G}$:

- $\mathcal{K} =$ so(3) $\oplus$ so(2), generators: $X_{AB} \ : \ (A, B) \in \{1, 2, 3\}, \{0, 4\}$, maximal compact subalgebra;

- $\mathcal{Q}$, generators: $X_{AB} \ : \ A \in \{1, 2, 3\}, B \in \{0, 4\}$, non-compact completion of $\mathcal{K}$;

- $\mathcal{A} = so(1, 1)$, generator: $D \doteq X_{34}$, dilatations;

- $\mathcal{M} = so(2, 1)$, generators: $X_{AB} \ : \ (A, B) \in \{0, 1, 2\}$, Lorentz subalgebra;

- $\mathcal{N}$, generators: $T_\mu = X_{\mu 3} + X_{\mu 4}$, $\mu = 0, 1, 2$, translations;

- $\widetilde{\mathcal{N}}$, generators: $C_\mu = X_{\mu 3} - X_{\mu 4}$, $\mu = 0, 1, 2$, special conformal transformations.

- $\mathcal{H}$, generators: D, X_{12}, Cartan subalgebra of $\mathcal{G}$;

Thus, we have several decompositions:

- $\mathcal{G} = \mathcal{K} \oplus \mathcal{Q}$, Cartan decomposition;

- $\mathcal{G} = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N}$, and $\mathcal{N} \to \widetilde{\mathcal{N}}$, Iwasawa decomposition;

- $\mathcal{G} = \mathcal{N} \oplus \mathcal{M} \oplus \mathcal{A} \oplus \widetilde{\mathcal{N}}$, Bruhat decomposition;

The subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \widetilde{\mathcal{N}}$ is a maximal parabolic subalgebra of $\mathcal{G}$.

Finally, we introduce the corresponding connected Lie groups:
$G = SO_0(3,2)$ with Lie algebra $\mathcal{G} = \mathrm{so}(3,2)$,
$K = SO(3) \times SO(2)$ is the maximal compact subgroup of G,
$A = \exp(\mathcal{A}) = SO_0(1,1)$ is abelian simply connected,
$N = \exp(\mathcal{N}) \cong \widetilde{N} = \exp(\widetilde{\mathcal{N}})$, are abelian simply connected subgroups of G preserved by the action of A.
The group $M \cong SO_0(2,1)$ (with Lie algebra $\mathcal{M}$) commutes with A.
The subgroup $P = MAN$ is a *maximal parabolic subgroup* of G. Parabolic subgroups are important because the representations induced from them generate all admissible irreducible representations of semisimple groups [Langlands,Knapp-Zuckerman].

Elementary representations

We use the approach of [D] which we adapt in a condensed form here. We work with so-called *elementary representations* (ERs). They are induced from representations of $P = MAN$, where we use finite-dimensional representations of spin $\mathsf{s} \in \frac{1}{2}\mathbb{Z}_+$ of M, (non-unitary) characters of A represented by the conformal weight Δ, and the factor N is represented trivially. The data s,Δ is enough to determine a weight $\Lambda \in \mathcal{H}^*$. Thus, we shall denote the ERs by C^Λ. Sometimes we shall write: $\Lambda = [\mathsf{s},\Delta]$. The representation spaces are C^∞ functions on G/P, or equivalently, on the locally isomorphic group $\widetilde{N}$ with appropriate asymptotic conditions (which we do not need explicitly). We recall that $\widetilde{N}$ is isomorphic to 3D Minkowski space-time $\mathfrak{M}$ whose elements will be denoted by $x = (x_0, x_1, x_2)$, while the corresponding elements of $\widetilde{N}$ will be denoted by $\tilde{n}_x$. The

Lorentzian inner product in $\mathfrak{M}$ is defined as usual:

$$\langle x, x' \rangle \doteq x_\mu x'^\mu = x_0 x'_0 - x_1 x'_1 - x_2 x'_2 \ , \qquad (4)$$

and we use the notation $\boldsymbol{x}^2 = \langle x, x \rangle$.

The representation action is given as follows:

$$(T^\Lambda(g)\varphi)(x) \ = \ y^{-\Delta} D^s(m)\, \varphi(x') \qquad (5)$$

the various factors being defined from the local Bruhat decomposition $G \cong_{\mathrm{loc}} \widetilde{N}AMN$:

$$g^{-1}\,\tilde{n}_x \ = \ \tilde{n}_{x'}\, a^{-1} m^{-1} n^{-1} \ , \qquad (6)$$

where $y \in \mathbb{R}_+$ parametrizes the elements $a \in A$, $m \in M$, $D^s(m)$ denotes the representation action of M, $n \in N$.

In the above general definition $\varphi(x)$ are considered as elements of the finite-dimensional representation space V^s in which act the operators $D^s(m)$. Following [DMPPT,D] we use scalar functions over an extended space $\mathfrak{M} \times \mathfrak{M}_0$, where $\mathfrak{M}_0$ is a cone parametrized by the variable $\zeta = (\zeta_0, \zeta_1, \zeta_2)$ subject to the condition:

$$\boldsymbol{\zeta}^2 = \langle \zeta, \zeta \rangle = \zeta_0^2 - \zeta_1^2 - \zeta_2^2 = 0.$$

The internal variable ζ will carry the representation D^s.

The functions on the extended space will be denoted as $\varphi(x, \zeta)$.† On these functions the infinitesimal action of our representations looks

†In mathematical language, we have passed to line-bundle formulation.

as follows:

$$
\begin{aligned}
&T_\mu = \partial_\mu, \quad \partial_\mu \doteq \frac{\partial}{\partial x_\mu}, \quad \mu = 0,1,2 \\
&D = -\sum_{\mu=0}^{2} x_\mu \partial_\mu - \Delta, \\
&X_{01} = x_0\partial_1 + x_1\partial_0 + \mathfrak{s}_{01}, \\
&X_{02} = x_0\partial_2 + x_2\partial_0 + \mathfrak{s}_{02}, \\
&X_{12} = -x_1\partial_2 + x_2\partial_1 + \mathfrak{s}_{12}, \\
&C_0 = 2x_0 D + \boldsymbol{x}^2\partial_0 - 2(x_1\mathfrak{s}_{01} + x_2\mathfrak{s}_{02}), \\
&C_1 = -2x_1 D + \boldsymbol{x}^2\partial_1 + 2(x_0\mathfrak{s}_{01} - x_2\mathfrak{s}_{12}), \\
&C_2 = -2x_2 D + \boldsymbol{x}^2\partial_2 + 2(x_0\mathfrak{s}_{02} + x_1\mathfrak{s}_{12}).
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
&\mathfrak{s}_{01} = \zeta_0 \frac{\partial}{\partial \zeta_1} + \zeta_1 \frac{\partial}{\partial \zeta_0}, \qquad \mathfrak{s}_{02} = \zeta_0 \frac{\partial}{\partial \zeta_2} + \zeta_2 \frac{\partial}{\partial \zeta_0}, \\
&\mathfrak{s}_{12} = -\zeta_1 \frac{\partial}{\partial \zeta_2} + \zeta_2 \frac{\partial}{\partial \zeta_1},
\end{aligned} \tag{8}
$$

and they satisfy the $\mathcal{M} = so(1,2)$ commutation relations

$$
\begin{aligned}
&[\mathfrak{s}_{01}, \mathfrak{s}_{02}] = -\mathfrak{s}_{12}, \qquad [\mathfrak{s}_{02}, \mathfrak{s}_{12}] = \mathfrak{s}_{01}, \\
&[\mathfrak{s}_{12}, \mathfrak{s}_{01}] = \mathfrak{s}_{02} \ .
\end{aligned} \tag{9}
$$

The Casimir of $\mathcal{G}$ is given by:

$$\mathcal{C} \;=\; \frac{1}{2} X_{AB} X^{AB} = -X_{01}^2 - X_{02}^2 + X_{12}^2 - D^2 \\ -3D - C_0 T_0 + C_1 T_1 + C_2 T_2 \qquad (10)$$

and it is constant on our representation C^{Λ} :

$$\mathcal{C}\,\varphi \;=\; -(\Delta(\Delta-3)+s(s+1))\,\varphi \;=\; \lambda(s,\Delta)\,\varphi\,. \qquad (11)$$

Note that the constant $\lambda(s,\Delta)$ has the same value if we replace Δ by $3-\Delta$. This means that the two boundary fields with conformal weights Δ and $3-\Delta$ are related, or in mathematical language, that the corresponding representations are (partially) equivalent.

Bulk representations

It is well known that the group $SO(3,2)$ is called also anti de Sitter group, as it is the group of isometry of 4D anti de Sitter space:

$$\xi^A \xi^B \eta_{AB} = 1 . \qquad (12)$$

There are several ways to parametrize anti de Sitter space. We shall utilize the same local Bruhat decomposition that we used in the previous subsection. Thus, we use the local coordinates on the factor-space $G/MN \cong_{\mathrm{loc}} \widetilde{N}A$, i.e., the coordinates $(x, y) = (x_0, x_1, x_2, y)$, $y \in \mathbb{R}_+$. In this setting the latter space is called *bulk* space, while 3D Minkowski space-time is called *boundary* space, as it is identified with the bulk boundary value $y = 0$. The functions on the bulk extended with the cone $\mathfrak{M}_0$ will be denoted by $\phi(x, y, \zeta)$.

Now we give a vector-field realization of so(3,2) on the bulk functions $\phi(x, y, \zeta)$:

$$
\begin{aligned}
&T_\mu = \partial_\mu, \quad \mu = 0,1,2 \\
&D = -\sum_{\mu=0}^{2} x_\mu \partial_\mu - y\partial_y, \\
&X_{01} = x_0\partial_1 + x_1\partial_0 + \mathfrak{s}_{01}, \\
&X_{02} = x_0\partial_2 + x_2\partial_0 + \mathfrak{s}_{02}, \\
&X_{12} = -x_1\partial_2 + x_2\partial_1 + \mathfrak{s}_{12}, \\
&C_0 = 2x_0 D + (\boldsymbol{x}^2 + y^2)\partial_0 + \\
&\quad + 2(y\mathfrak{s}_{12} - x_1\mathfrak{s}_{01} - x_2\mathfrak{s}_{02}), \\
&C_1 = -2x_1 D + (\boldsymbol{x}^2 + y^2)\partial_1 + \\
&\quad + 2(y\mathfrak{s}_{02} + x_0\mathfrak{s}_{01} - x_2\mathfrak{s}_{12}), \\
&C_2 = -2x_2 D + (\boldsymbol{x}^2 + y^2)\partial_2 + \\
&\quad + 2(-y\mathfrak{s}_{01} + x_0\mathfrak{s}_{02} + x_1\mathfrak{s}_{12}),
\end{aligned}
\tag{13}
$$

Note that the realization of so(3,2) on the boundary given in (7) may be obtained from (13) by replacing $y\partial_y \to \Delta$ and then taking the limit $y \to 0$.

The realization (13) causes the infinitesimal transformation of the bulk coordinates:

$$
\begin{aligned}
T_\mu &: \; x_\mu \to x_\mu + a, \\
D &: \; x_\mu \to (1-a)x_\mu, \qquad y \to (1-a)y, \\
X_{0\mu} &: \; x_0 \to x_0 + a x_\mu, \quad x_\mu \to x_\mu + a x_0, \;\; \mu = 1,2 \\
X_{12} &: \; x_1 \to x_1 + a x_2, \qquad x_2 \to x_2 - a x_1, \\
C_0 &: \; x_0 \to x_0 + a(y^2 - x_0^2 - x_1^2 - x_2^2), \\
&\quad\; x_{1,2} \to (1 - 2a x_0) x_{1,2}, \quad y \to (1 - 2a x_0) y, \\
C_1 &: \; x_1 \to x_1 + a(y^2 + x_0^2 + x_1^2 - x_2^2), \\
&\quad\; x_{0,2} \to (1 + 2a x_1) x_{0,2}, \quad y \to (1 + 2a x_1) y, \\
C_2 &: \; x_2 \to x_2 + a(y^2 + x_0^2 - x_1^2 + x_2^2), \\
&\quad\; x_{0,1} \to (1 + 2a x_2) x_{0,1}, \quad y \to (1 + 2a x_2) y
\end{aligned}
$$

It follows that every $SO(3,2)$ invariant of the two points (x_μ, y) and (x'_μ, y') is a function of

$$u \;=\; \frac{4yy'}{(\boldsymbol{x}-\boldsymbol{x'})^2+(y+y')^2}\ . \qquad (14)$$

We shall use also one-point $SO(3,2)$ invariants - functions of the variable

$$\widehat{u} \;=\; \frac{4y}{\boldsymbol{x}^2+(y+1)^2}\ . \qquad (15)$$

obtained from u by setting $x'_\mu = 0,\ y' = 1$.

Eigenvalue problem and two-point functions in the bulk

Eigenvalue problem of Casimir in the bulk

As we explained in the Introduction we first concentrate on the holography principle, or boundary-to-bulk correspondence, which means to have an operator which maps a boundary field φ to a bulk field ϕ. This map must be invariant w.r.t. the Lie algebra $so(3,2)$. In particular, this means that the Casimir must have the same values in the boundary and bulk representations. The Casimir on the boundary representation C^{Λ} is a constant $\lambda(s,\Delta)$ given in (11). Clearly, the equivalent bulk representation will consist only of functions on which the Casimir has the same value.

Explicitly, the Casimir operator is realized in the bulk as follows

$$
\begin{aligned}
\mathcal{C} &= \mathcal{C}_B + \mathcal{C}_I - 2y(\mathfrak{s}_{12}\partial_0 - \mathfrak{s}_{02}\partial_1 + \mathfrak{s}_{01}\partial_2), \\
\mathcal{C}_B &= y^2(-\partial_0^2 + \partial_1^2 + \partial_2^2) - y^2\partial_y^2 + 2y\partial_y \ , \\
\mathcal{C}_I &= (-\mathfrak{s}_{01}^2 - \mathfrak{s}_{02}^2 + \mathfrak{s}_{12}^2) \qquad (16)
\end{aligned}
$$

where $\mathcal{C}_I$ is the Casimir operator of $so(1,2)$ in terms of the internal variables.

Since the Casimir in the bulk is not constant but a differential operator our representation functions will be found as the Casimir eigenfunctions in the bulk. Thus, we consider the eigenvalue problem of the Casimir operator of so(3,2):

$$
\mathcal{C}\Psi(x,y,\zeta) = \lambda(s,\Delta)\Psi(x,y,\zeta) \ . \qquad (17)
$$

Having in mind the degeneracy of Casimir values for partially equivalent representations

($\Delta \leftrightarrow 3-\Delta$) we shall need also the appropriate asymptotic condition:

$$\phi(x,y,\zeta)|_{y\to 0} \quad \longrightarrow \quad y^\alpha \phi(x,0,\zeta) \ , \qquad \alpha = \Delta, 3-\Delta \ . \tag{18}$$

Below we shall use the asymptotic condition $\alpha = \Delta$, while the other choice will be implemented by equivalence.

One may write $\mathcal{C}_B$ in terms of only $\hat{u}$:

$$\mathcal{C}_B = \hat{u}^2(\hat{u}-1)\frac{d^2}{d\hat{u}^2} + 2\hat{u}\frac{d}{d\hat{u}} \ . \tag{19}$$

We are interested in solutions in which the ζ-dependence is factored out in the form

$$\Psi = \psi(x,y)\, Q(x,y,\zeta)^s \ , \qquad s \in \mathbb{Z}_+$$

We assume that ψ is a $SO(3,2)$ invariant, thus it is function only of $\hat{u}$: $\psi(x,y) = \psi(\hat{u})$. Furthermore we require that Q is an eigenfunction of $\mathcal{C}_I$. It follows that Ψ is also an eigenfunction of $\mathcal{C}_I$:

$$\mathcal{C}_I \Psi = \lambda_I \Psi = -s(s+1)\Psi \ . \tag{20}$$

With the fixed vector $(\zeta'_0, \zeta'_1, \zeta'_2)$ in the internal space, Q is given by

$$
\begin{aligned}
Q \;=\; & \frac{2\, I_1 - (\boldsymbol{x}^2 - (y+1)^2)\, I_2 - 2(y+1)\, I_3}{\boldsymbol{x}^2 + (y+1)^2} \;, \\
& I_1 = \langle x, \zeta \rangle \, \langle x, \zeta' \rangle , \qquad I_2 = \langle \zeta, \zeta' \rangle , \\
& I_3 = \sum_{\mu=0}^{2} x_\mu (\zeta \times \zeta')_\mu \;, \qquad (21)
\end{aligned}
$$

where $\zeta \times \zeta'$ is the standard vector product.

With this form of Q the eigenvalue problem is reduced to the second order differential equation:

$$
\begin{aligned}
& \left((\widehat{u} - 1)\widehat{u}^2 \frac{d^2}{d\widehat{u}^2} + 2\widehat{u} \frac{d}{d\widehat{u}} - s(s+1)\widehat{u} \right) \psi(\widehat{u}) \\
& = (\lambda - \lambda_I)\psi(\widehat{u}) = \Delta(3 - \Delta)\psi(\widehat{u}) \qquad (22)
\end{aligned}
$$

Two-point Green function in bulk

We need also the two-point Green function in bulk. Standardly for this we derive the Green function of the operator $\mathcal{C} - \lambda$

$$(\mathcal{C}-\lambda)\mathsf{G}(x,y,\zeta;x',y',\zeta') = y^4\delta^3(x-x')\delta(y-y')(\zeta,\zeta')^s. \tag{23}$$

The computation of G is more or less same as the ones for eigenvalue problem of $\mathcal{C}$ in the previous subsection. We assume G has a factored form

$$\mathsf{G}(x,y,\zeta;x',y',\zeta') = f(u)\,Q(x,y,\zeta;x',y',\zeta')^s,$$

where u is the $SO(3,2)$ invariant of two points (x,y) and (x',y') given in (14).

Then G is given by

$$\mathsf{G} = u^{\Delta} F(u)\, Q^s \ , \tag{24}$$

$$Q = \frac{2I'_1 - ((\boldsymbol{x} - \boldsymbol{x}')^2 - (y+y')^2)I_2 - 2(y+y')I'_3}{(\boldsymbol{x} - \boldsymbol{x}')^2 + (y+y')^2}$$

$$I'_1 = \langle x - x', \zeta\rangle \, \langle x - x', \zeta'\rangle, \qquad I_2 = \langle \zeta, \zeta'\rangle,$$

$$I'_3 = \sum_{\mu=0}^{2} (x_\mu - x'_\mu)(\zeta \times \zeta')_\mu,$$

and $F(u)$ is a singular solution of the hypergeometric equation

$$\Big(u(1-u)\frac{d^2}{du^2} + 2[\Delta - 1 - \Delta\, u]\frac{d}{du} \ + \\ + \ (s - \Delta + 1)(s + \Delta)\Big) F(u) = 0 \tag{25}$$

By another calculation we obtain:

$$(\mathcal{C} - \lambda)\mathsf{G} = -Q^s u^{\Delta+1}\Big(u(1-u)\frac{d^2}{du^2} + \\ +2[\Delta - 1 - \Delta\, u]\frac{d}{du} + (s - \Delta + 1)(s + \Delta)\Big) F(u) \tag{26}$$

Thus if $F(u)$ is a singular solution of the hypergeometric equation (25) then we obtain the RHS of (23). The delta functions in the RHS corresponds to the singularity at $u = 1 \Leftrightarrow x_\mu = x'_\mu,\ y = y'$.

Bulk-boundary correspondence

Let $\varphi \in C^{\Lambda}$ and ϕ be fields on the boundary and in the bulk, respectively. We assume that they are eigenfunctions of the Casimir operator with the same eigenvalue:

$$\mathcal{C}\varphi = \lambda\varphi, \qquad \mathcal{C}\phi = \lambda\phi. \tag{27}$$

The bulk field behaves as in (18) when approaching the boundary. We choose $\alpha = \Delta$ and define the bulk-to-boundary operator L_{Δ} by

$$\varphi(x,\zeta) = (L_{\Delta}\phi)(x,\zeta) := \lim_{y\to 0} y^{-\Delta}\phi(x,y,\zeta). \tag{28}$$

On the other hand the boundary-to-bulk operator $\tilde{L}_{\Lambda}$ is defined by

$$\phi(x,y,\zeta) = (\tilde{L}_{\Lambda}\varphi)(x,\zeta) := \tag{29}$$
$$= \int S_{\Lambda}(x-x',y;\zeta,\partial_{\zeta'})\varphi(x',\zeta')d^3x'$$

where the kernel S_Λ is obtained from the bulk two-point Green function G by sending one point to the boundary:

$$S_\Lambda(x-x', y; \zeta, \partial_{\zeta'}) = \lim_{y'\to 0} y'^{\Delta-3} \mathsf{G}(x, y, \zeta; x', y', \partial_{\zeta'}). \tag{30}$$

The formula for S_Λ is given by

$$S_\Lambda \;=\; N_\Lambda \tilde{u}^{3-\Delta} R^s, \quad \tilde{u} = \frac{4y}{(\boldsymbol{x}-\boldsymbol{x}')^2 + y^2},$$
$$R = \frac{\tilde{u}\mathcal{L}}{4y}, \tag{31}$$

with

$$\mathcal{L} = 2\tilde{I}_1 - ((\boldsymbol{x}-\boldsymbol{x}')^2 - y^2)\tilde{I}_2 - 2y\tilde{I}_3, \tag{32}$$
$$\tilde{I}_1 = \langle x - x', \zeta\rangle \langle x - x', \partial_{\zeta'}\rangle, \quad \tilde{I}_2 = \langle \zeta, \partial_{\zeta'}\rangle,$$
$$\tilde{I}_3 = \sum_{\mu=0}^{2} (x_\mu - x'_\mu)(\zeta \times \partial_{\zeta'})_\mu \;,$$

and N_Λ is a normalization constant depending on the representation $\Lambda = [\mathrm{s}, \Delta]$.

Now we check consistency of the operators

$L_\triangle$ and $\tilde{L}_\Lambda$:

$$L_\triangle \circ \tilde{L}_\Lambda = \mathbf{1}_\Lambda \doteq \mathbf{1}_{C^\Lambda}\,, \qquad \tilde{L}_\Lambda \circ L_\triangle = \mathbf{1}_{\text{bulk}}\,. \tag{33}$$

For the first relation in (33) we have to show that:

$$\varphi(x,\zeta) = (L_\triangle \circ \tilde{L}_\Lambda \varphi)(x,\zeta) = \tag{34}$$
$$= \lim_{y\to 0} y^{-\triangle} \int S_\Lambda(x-x', y; \zeta, \partial_{\zeta'})\varphi(x',\zeta')d^3x'$$

We take the limit first by exchanging it and the integral. To calculate the limit it is necessary to express the kernel S_Λ in another form. To this end we establish the following formula of Fourier transform:

$$\int \frac{e^{i\langle p,X\rangle}}{(\langle X,X\rangle + y^2)^\alpha}\frac{d^3X}{(2\pi)^{3/2}} = \frac{i\pi}{(-1)^{2\alpha-1}2^\alpha\Gamma(\alpha)}$$
$$\times \left(\frac{\sqrt{-\boldsymbol{p}^2}}{y}\right)^{\alpha-3/2} H^{(1)}_{\alpha-3/2}(y\sqrt{-\boldsymbol{p}^2}\,), \tag{35}$$

where $X_\mu = x_\mu - x'_\mu$ and $H^{(1)}_\beta$ is a Hankel function. The (X_1, X_2) integration can by carried out by making use of the following two formulae. First one is a formula for $(d-1)$ dimensional angular integration in d dimensional Euclidean space:

$$\int f(r) e^{-i\vec{p}\cdot\vec{x}} \frac{d^d x}{(2\pi)^d} = \tag{36}$$

$$\left(\frac{1}{2\pi}\right)^{d/2} \int_0^\infty r f(r) \left(\frac{r}{p}\right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(pr)\, dr \ ,$$

$$\vec{p} = (p_1, p_2, \cdots, p_d),\ \vec{x} = (x_1, x_2, \cdots, x_d),$$

$$p^2 = \sum_{k=1}^{d} p_k^2,\ r^2 = \sum_{k=1}^{d} x_i^2,$$

which is valid for any radial function $f(r)$. Second formula is an integration of Bessel function:

$$\int_0^\infty \frac{r^{\beta+1} J_\beta(ar)}{(r^2+\rho^2)^{\gamma+1}} dr = \frac{a^\gamma \rho^{\beta-\gamma} K_{\beta-\gamma}(a\rho)}{2^\gamma \Gamma(\gamma+1)} \tag{37}$$

$$2\Re\gamma + \frac{3}{2} > \Re\beta > -1.$$

We modify the second formula (38). Set $\beta = 0$ and replace ρ with $-i\rho$, then use the relation between Bessel functions

$$K_\gamma(z) = \frac{\pi}{2} i e^{\gamma\pi i/2} H_\gamma^{(1)}(iz), \quad -\pi < \arg z < \frac{\pi}{2} \tag{38}$$

we obtain

$$\int_0^\infty \frac{r J_0(ar)}{(r^2-\rho^2)^{\gamma+1}} dr = \tag{39}$$
$$= \frac{i\pi a^\gamma}{(-1)^\gamma 2^{\gamma+1}\Gamma(\gamma+1)\rho^\gamma} H_\gamma^{(1)}(a\rho)$$

Now we return to the Fourier transform (35). Angular integration in $X_1 X_2$ plane is performed by (37) and we use (40) for the radial integration in the plane:

$$\int \frac{e^{i\langle p,X\rangle}}{(\langle X,X\rangle + y^2)^\alpha} \frac{d^3X}{(2\pi)^{3/2}} =$$

$$\int_{-\infty}^{\infty} \frac{dX_0}{\sqrt{2\pi}} \int_0^\infty \frac{dr\ rJ_0(\tilde{p}r)}{(-1)^\alpha (r^2 - X_0^2 - y^2)^\alpha} e^{ip_0 X_0}$$

$$= \frac{i\pi}{(-1)^{2\alpha-1}\Gamma(\alpha)} \left(\frac{\tilde{p}}{2}\right)^{\alpha-1} \times$$

$$\times \int_0^\infty \frac{dX_0}{\sqrt{2\pi}} \frac{H^{(1)}_{\alpha-1}(\tilde{p}\sqrt{X_0^2+y^2}\,)}{(X_0^2+y^2)^{(\alpha-1)/2}} \cos p_0 X_0$$

where

$$r^2 = X_1^2 + X_2^2, \qquad \tilde{p}^2 = p_1^2 + p_2^2.$$

Recalling that

$$H^{(1)}_\beta(z) = J_\beta(z) + iY_\beta(z)$$

X_0 integration is performed by the formulae of Fourier cosine transform [Bateman]

$f(r)$	$\int_0^\infty f(r)\cos(r\rho)\,dr$
$\dfrac{J_\beta(a\sqrt{r^2+b^2}\,)}{(r^2+b^2)^{\beta/2}}$	$\begin{cases} \sqrt{\dfrac{\pi b}{2}}\dfrac{(a^2-\rho^2)^{\beta/2-1/4}}{(ab)^\beta}\times \\ \times J_{\beta-1/2}(b\sqrt{a^2-\rho^2}\,) & 0<\rho<a \\ 0 & a<\rho \end{cases}$
$\dfrac{Y_\beta(a\sqrt{r^2+b^2}\,)}{(r^2+b^2)^{\beta/2}}$	$\begin{cases} \sqrt{\dfrac{\pi b}{2}}\dfrac{(a^2-\rho^2)^{\beta/2-1/4}}{(ab)^\beta}\times \\ \times Y_{\beta-1/2}(b\sqrt{a^2-\rho^2}\,) & 0<\rho<a \\ -\sqrt{\dfrac{2b}{\pi}}\dfrac{(\rho^2-a^2)^{\beta/2-1/4}}{(ab)^\beta}\times \\ \times K_{\beta-1/2}(b\sqrt{\rho^2-a^2}\,) & a<\rho \end{cases}$

$$\Re\,\beta > -\tfrac{1}{2}, \quad a, b > 0$$

By these formula we obtain

$$\int_0^\infty dX_0 \frac{H^{(1)}_{\alpha-1}(\tilde{p}\sqrt{X_0^2+y^2}\,)}{(X_0^2+y^2)^{(\alpha-1)/2}} \cos(p_0 X_0) =$$

$$= \begin{cases} \left(\frac{\pi y}{2}\right)^{1/2} \frac{(-\langle p,p\rangle)^{\alpha/2-3/4}}{(\tilde{p}y)^{\alpha-1}} \times \\ \times H^{(1)}_{\alpha-3/2}(y\sqrt{\tilde{p}^2-p_0^2}\,) & 0 < p_0 < \tilde{p} \\ -i\left(\frac{2y}{\pi}\right)^{1/2} \frac{\langle p,p\rangle^{\alpha/2-3/4}}{(\tilde{p}y)^{\alpha-1}} \times \\ \times K_{\alpha-3/2}(y\sqrt{p_0^2-\tilde{p}^2}\,) & \tilde{p} < p_0 \end{cases}$$

$$= \left(\frac{\pi y}{2}\right)^{1/2} \frac{(-\langle p,p\rangle)^{\alpha/2-3/4}}{(\tilde{p}y)^{\alpha-1}} \times \\ \times\ H^{(1)}_{\alpha-3/2}(y\sqrt{\tilde{p}^2-p_0^2}\,) \qquad (40)$$

In the last equality the relation (38) was used to unify two separate cases. Note that $\langle p,p\rangle = p_0^2-\tilde{p}^2$. In this way the Fourier transform (35) has been established.

Now we evaluate the Fourier transform of the kernel S_Λ

$$\int S_\Lambda(X,y;\zeta,\partial_{\zeta'})e^{i\langle p,X\rangle}\frac{d^3X}{(2\pi)^{3/2}} = \tag{41}$$
$$= N_\Lambda \int \frac{(4y)^{3-\Delta}\ e^{i\langle p,X\rangle}}{(\langle X,X\rangle+y^2)^{s-\Delta+3}}\ \boldsymbol{S}^s\ \frac{d^3X}{(2\pi)^{3/2}} =$$
$$= \frac{-i\pi N_\Lambda}{2^{s+\Delta-1}\Gamma(s-\Delta+3)y^{s-3/2}}\boldsymbol{S}^s \times$$
$$\times(\sqrt{-\boldsymbol{p}^2}\,)^{s-\Delta+3/2}H^{(1)}_{s-\Delta+3/2}(y\sqrt{-\boldsymbol{p}^2}\,),$$

where

$$\boldsymbol{S} \ = \ -2(\partial_p\cdot\zeta)\,\partial_p\cdot\partial_{\zeta'}+(\langle\partial_p,\partial_p\rangle+y^2)\langle\zeta,\partial_{\zeta'}\rangle + \ 2iy\langle\partial_p,\zeta\times\partial_{\zeta'}\rangle, \tag{42}$$

with $a\cdot b=\sum_{\mu=0}^{2} a_\mu b_\mu$. Inverse Fourier transform gives the following formula of the kernel

$$S_\Lambda = \frac{-i\pi N_\Lambda}{2^{s+\Delta-1}\Gamma(s-\Delta+3)y^{s-3/2}}\times \tag{43}$$
$$\times\int \boldsymbol{S}^s(\sqrt{-\boldsymbol{p}^2}\,)^{s-\Delta+3/2}H^{(1)}_{s-\Delta+3/2}(y\sqrt{-\boldsymbol{p}^2}\,)$$
$$\times\ e^{-i(\langle p,X\rangle}\frac{d^3p}{(2\pi)^{3/2}}$$

Since we take a limit of $y \to 0$, we replace the Hankel function with its asymptotic form

$$-iH_{\alpha}^{(1)}(z) \to -\frac{\Gamma(\alpha)}{\pi}\left(\frac{2}{z}\right)^{\alpha}, \qquad z \to 0$$

Then

$$\begin{aligned} &-i(\sqrt{-\boldsymbol{p}^2}\,)^{s-\Delta+3/2} H_{s-\Delta+3/2}^{(1)}(y\sqrt{-\boldsymbol{p}^2}\,) = \\ &= -\frac{\Gamma(s-\Delta+3/2)}{\pi}\left(\frac{2}{y}\right)^{s-\Delta+3/2}, \\ &\qquad s-\Delta+3/2 \notin \mathbb{Z}_- \ , \end{aligned} \tag{44}$$

is independent of p_μ so that the action of $\boldsymbol{S}$ is reduced to $y^2\langle\zeta,\partial_{\zeta'}\rangle$ and the integration over p becomes Dirac's delta function:

$$\begin{aligned} S_\Lambda \to -\frac{(2\pi)^{3/2} N_\Lambda \Gamma(s-\Delta+3/2)}{2^{2\Delta-5/2}\Gamma(s-\Delta+3)} \times \\ \times\ y^{\Delta}\delta^3(X)\langle\zeta,\partial_{\zeta'}\rangle^s \\ s-\Delta+3/2 \notin \mathbb{Z}_- \ , \quad y \to 0 \end{aligned} \tag{45}$$

Substituting this formula of S in (35) we obtain:

$$\varphi(x,\zeta) = -\frac{\pi^{3/2} N_\Lambda \Gamma(s-\Delta+3/2)}{2^{2\Delta-4}\Gamma(s-\Delta+3)}\varphi(x,\zeta)$$
$$s-\Delta+3/2 \notin \mathbb{Z}_- \,, \; s-\Delta+3 \notin \mathbb{Z}_- \quad (46)$$

From the latter we see the first consistency relation (33) being true by an appropriate choice of N_Λ, e.g.

$$N_\Lambda \quad = \quad -\frac{2^{2\Delta-4}\,\Gamma(s-\Delta+3)}{\pi^{3/2}\,\Gamma(s-\Delta+3/2)} \,, \qquad (47)$$
$$s-\Delta+3/2 \notin \mathbb{Z}_- \,, \quad s-\Delta+3 \notin \mathbb{Z}_-$$

As a *Corollary* we conclude that for generic values of Δ we can reconstruct a function on anti de Sitter space from its boundary value. Indeed, suppose we have:

$$\phi(x,y,\zeta) \; = \; \int S_\Lambda(x-x',y;\zeta,\partial_{\zeta'}) f(x',\zeta')\, dx' \; . \qquad (48)$$

Then we have for the boundary value:

$$\psi_0(x,\zeta) \doteq (L_\Delta \phi)(x,\zeta) = \lim_{y\to 0} \; y^{-\Delta} \; \phi(x,y,\zeta)$$
$$= \lim_{y\to 0} \; y^{-\Delta} \; \int S_\Lambda(x-x',y;\zeta,\partial_{\zeta'}) f(x',\zeta') dx'$$
$$= f(x) \tag{49}$$

Now we can prove the second consistency relation in (33):

$$\left(\tilde{L}_\Lambda \circ L_\Delta \phi\right)(x,y,\zeta) = \tag{50}$$
$$= \int S_\Lambda(x-x',y;\zeta,\partial_{\zeta'}) \left(L_\Delta \phi\right)(x',\zeta') dx' =$$
$$= \int S_\Lambda(x-x',y;\zeta,\partial_{\zeta'}) \lim_{y'\to 0} \; y'^{-\Delta} \phi(x',y',\zeta') =$$
$$= \int S_\Lambda(x-x',y;\zeta,\partial_{\zeta'}) \psi_0(x',\zeta') dx' = \phi(x,y,\zeta)$$

where in the last line we used (49).

Intertwining properties

Here we investigate the intertwining properties of the boundary $\leftrightarrow$ bulk operators.

Bulk-to-boundary operator L_Δ

It is not difficult to verify the intertwining property of the representations of $so(3,2)$ on the boundary and in the bulk. Namely, one may verify the following by direct computation:

$$\tilde{X} \circ L_\Delta = L_\Delta \circ \hat{X} \ , \quad X \in so(3,2), \qquad (51)$$

where $\tilde{X}$ denotes the action of the generator X on the boundary (7) and $\hat{X}$ denotes the action of the generator in the bulk (13). More explicitly,

$$\tilde{X}\varphi(x_\mu, \zeta_\mu) = \lim_{y\to 0} y^{-\Delta} \hat{X}\phi(x_\mu, y, \zeta_\mu) \qquad (52)$$

If the field φ belongs to the conjugate representation $\varphi \in C^{\tilde{\Lambda}}$, $\tilde{\Lambda} = [s, 3-\Delta]$, then relations (51),(52) hold with the change $\Delta \to 3-\Delta$, the same change being made also in (7).

Boundary-to-bulk operator $\tilde{L}_\Lambda$

The intertwining property means that

$$\hat{X} \circ \tilde{L}_\Lambda = \tilde{L}_\Lambda \circ \tilde{X}. \tag{53}$$

More explicitly, it reads

$$\begin{aligned} &\hat{X}\,\phi(x_\mu, y, \zeta_\mu) = \\ &= \int S_\Lambda(x_\mu, y, \zeta_\mu; x'_\mu, \partial_{\zeta'_\mu})\,\tilde{X}_\Lambda\,\varphi(x'_\mu, \zeta'_\mu) d^3x' \end{aligned} \tag{54}$$

This is an immediate consequence of $L_\Delta \circ \tilde{L}_\Lambda = \mathbf{1}_\Lambda$, $\tilde{L}_\Lambda \circ L_\Delta = \mathbf{1}_{\text{bulk}}$ and (51). By sandwiching (51) by $\tilde{L}_\Lambda$ one has

$$\tilde{L}_\Lambda \circ L_\Delta \circ \hat{X} \circ \tilde{L}_\Lambda = \tilde{L}_\Lambda \circ \tilde{X} \circ L_\Delta \circ \tilde{L}_\Lambda \,.$$

This is nothing but (53).

Further intertwining relations

We start by recording the second limit of the bulk functions

$$\varphi_0(x,\zeta) \doteq \lim_{y\to 0} \; y^{\Delta-3} \; \phi(x,y,\zeta) = \tag{55}$$

$$= \lim_{y\to 0} \; y^{\Delta-3} \int S_\Lambda(x-x',y;\zeta,\partial_{\zeta'})\psi_0(x',\zeta')dx' =$$

$$= \frac{\mathcal{N}_\Lambda}{\gamma_{\tilde{\Lambda}}} \int dx' \; G_{\tilde{\Lambda}}(x-x';\zeta,\partial_{\zeta'}) \; \psi_0(x',\zeta') \; , \tag{56}$$

$$\mathcal{N}_\Lambda = 4^{3-\Delta} N_\Lambda \; , \tag{57}$$

where we have recovered the well-known conformal two-point function:

$$G_\Lambda(x;\zeta,\zeta') \;=\; \gamma_\Lambda \; \frac{(r(x;\zeta,\zeta'))^s}{(\boldsymbol{x}^2)^\Delta} \; , \tag{58}$$

$$r(x;\zeta,\zeta') \;=\; r(x)_{\mu\sigma}\zeta^\mu\zeta'^\sigma \; , \tag{59}$$

$$r(x)_{\mu\sigma} = \frac{2}{\boldsymbol{x}^2} x_\mu x_\sigma - g_{\mu\sigma}$$

$$g \;=\; (g_{\mu\nu}) = \mathrm{diag}(1,-1,-1)$$

for the conjugate weight $\tilde{\Lambda} = [s, 3-\Delta]$. The latter is natural since $\psi_0 \in C^{\Lambda}$, $\varphi_0 \in C^{\tilde{\Lambda}}$, and the conformal two-point function realizes the equivalence of the conjugate representations Λ, $\tilde{\Lambda}$ which have the same Casimir values, as we have seen. The normalization constant γ_Λ depends on the representation $\Lambda = [s, \Delta]$ and below we derive a formula for the product $\gamma_\Lambda \gamma_{\tilde{\Lambda}}$.

Further, using (55) we define the operator G_Λ through the kernel $G_\Lambda(x; \zeta, \zeta')$:

$$\begin{aligned} G_\Lambda \quad &: \quad C^{\tilde{\Lambda}} \to C^{\Lambda} \ , \qquad (60) \\ (G_\Lambda \varphi_0)(x,\zeta) \ &= \ \int dx' \ G_\Lambda(x-x'; \zeta, \partial_{\zeta'}) \ \varphi_0(x',\zeta') \end{aligned}$$

Then relation (55) may be written as:

$$L_{\tilde{\Delta}} = \frac{\mathcal{N}_\Lambda}{\gamma_{\tilde{\Lambda}}} \, G_{\tilde{\Lambda}} \circ L_\Delta \ , \qquad \tilde{\Delta} \doteq 3 - \Delta \qquad (61)$$

as operators acting on the bulk functions $\phi(x,y,\zeta)$.

Note that at generic points (those not excluded in (46)) the operators G_{Λ} and $G_{\tilde{\Lambda}}$ are inverse to each other:

$$G_{\Lambda} \circ G_{\tilde{\Lambda}} = \mathbf{1}_{\Lambda} \ , \qquad G_{\tilde{\Lambda}} \circ G_{\Lambda} = \mathbf{1}_{\tilde{\Lambda}} \ . \tag{62}$$

At generic points from this we can obtain a lot of interesting relations, e.g., applying $\tilde{L}_{\Lambda}$ from the right we get:

$$L_{\tilde{\Delta}} \circ \tilde{L}_{\Lambda} = \frac{\mathcal{N}_{\Lambda}}{\gamma_{\tilde{\Lambda}}} \ G_{\tilde{\Lambda}} \tag{63}$$

Then we write down the conjugate relation:

$$L_{\Delta} \circ \tilde{L}_{\tilde{\Lambda}} = \frac{\mathcal{N}_{\tilde{\Lambda}}}{\gamma_{\Lambda}} \ G_{\Lambda} \tag{64}$$

Then we combine relations (63) and (64):

$$L_{\Delta} \circ \tilde{L}_{\tilde{\Lambda}} \circ L_{\tilde{\Delta}} \circ \tilde{L}_{\Lambda} = \frac{\mathcal{N}_{\tilde{\Lambda}}}{\gamma_{\Lambda}} \frac{\mathcal{N}_{\Lambda}}{\gamma_{\tilde{\Lambda}}} \, G_{\Lambda} \circ G_{\tilde{\Lambda}} = \frac{\mathcal{N}_{\tilde{\Lambda}}}{\gamma_{\Lambda}} \frac{\mathcal{N}_{\Lambda}}{\gamma_{\tilde{\Lambda}}} \, \mathbf{1}_{\Lambda} \tag{65}$$

For the LHS of (65) we use first the second relation of (33), then the first to obtain:

$$L_{\Delta} \circ \tilde{L}_{\tilde{\Lambda}} \circ L_{\tilde{\Delta}} \circ \tilde{L}_{\Lambda} = L_{\Delta} \circ \mathbf{1}_{\text{bulk}} \circ \tilde{L}_{\Lambda} = L_{\Delta} \circ \tilde{L}_{\Lambda} = \mathbf{1}_{\Lambda} \ . \tag{66}$$

Thus, from (65) and (66) using (47) and (57) follows:

$$\gamma_\Lambda \, \gamma_{\tilde\Lambda} = \mathcal{N}_\Lambda \, \mathcal{N}_{\tilde\Lambda} = \tag{67}$$
$$= \frac{2^4 \Gamma(s-\Delta+3)\,\Gamma(s+\Delta)}{\pi^3\,\Gamma(s-\Delta+3/2)\,\Gamma(s+\Delta-3/2)} \ ,$$
$$s-\Delta+3/2 \notin \mathbb{Z}_- \ , \quad s-\Delta+3 \notin \mathbb{Z}_- \ ,$$
$$s+\Delta-3/2 \notin \mathbb{Z}_- \ , \quad s+\Delta \notin \mathbb{Z}_- \ .$$

The product of constants in (67) should be proportional to the the analytic continuation of the Plancherel measure for the Plancherel formula contribution of the principal series of unitary irreps of G, cf., e.g., [D], but we shall not go into that.

Finally, we notice that all operators that we have used may be found on the following commutative diagram:

$$\mathbf{bulk}[\phi]$$

$$L_{\tilde{\Delta}} \;\swarrow\;\nearrow\; \tilde{L}_{\tilde{\Lambda}} \qquad\qquad \tilde{L}_{\Lambda} \;\nwarrow\;\searrow\; L_{\Delta}$$

$$C_{\tilde{\Lambda}}[\varphi_0] \qquad \underset{G_{\Lambda}}{\overset{G_{\tilde{\Lambda}}}{\rightleftarrows}} \qquad C_{\Lambda}[\psi_0]$$