

p -ADIC MATHEMATICAL PHYSICS AND ITS APPLICATIONS

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*8th MATHEMATICAL PHYSICS MEETING:
Summer School and Conference on Modern Mathematical
Physics*

24–31 August 2014

Belgrade, SERBIA

- 1 Introduction
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1. Introduction: p -adic mathematical physics

- This talk is related to p -adic numbers, adeles, their analysis and applications
- Application of p -adic numbers in modeling some physical systems started in 1987 by construction of p -adic scattering string amplitudes (Volovich, ...)
- p -Adic mathematical physics


1. Introduction: applications in various parts of modern mathematical physics and related fields

- p -adic and adelic string theory
- p -adic and adelic quantum mechanics and (quantum) field theory
- p -adic and adelic gravity and (quantum) cosmology
- p -adic and adelic space-time structure at the Planck scale
- p -adic and adelic dynamical systems
- p -adic stochastic processes
- p -adic aspects of information theory
- p -adic structure of the genetic code and protein dynamics
- p -adic wavelets
- spin glasses, p -adic unordered systems, ...

1. Introduction: international interdisciplinary journal on p -adics

Volume 1, Number 1
January–March 2009

ISSN: 2070-0466





**p-ADIC NUMBERS,
ULTRAMETRIC
ANALYSIS AND
APPLICATIONS**

Editor-in-Chief
Igor V. Volovich

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p-Adic Numbers, Ultrametric Analysis and Applications

ISSN: 2070-0466

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p-Adic Numbers, Ultrametric Analysis and Applications is available ONLINE by subscription from www.springerlink.com.

p-Adic Numbers, Ultrametric Analysis and Applications (ISSN: 2070-0466) (CODEN: PNAAPP) appears quarterly in published 4 issues per year by PLEIADES PUBLISHING, 27500, rue de l'Éclairage, 215 Spring House, 4th Floor, New York, NY 10013. Periodical postage paid at New York, NY and additional mailing offices. POSTMASTER: Send address changes to p-Adic Numbers, Ultrametric Analysis and Applications, Springer, 233 Spring Street, New York, NY 10013, USA.

Volume 1 (4 issues) published in 2009.

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TEL: 7-495-716-1000; FAX: 7-495-716-1000; E-MAIL: corresp@miik.ru; WEB: <http://www.miik.ru>

The journal was founded in 2008.
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1. Introduction: international conferences on p -adic mathematical physics



2nd International Conference on p -Adic Mathematical Physics
Belgrade, 2005

2. p -Adic numbers, adeles and their functions

- p -Adic numbers discovered by Kurt Hensel (1861-1941) in 1897.
- Any p -adic number ($x \in \mathbb{Q}_p$) has a unique canonical representation, ($p = 2, 3, 5, 7, 11, \dots =$ a prime number)

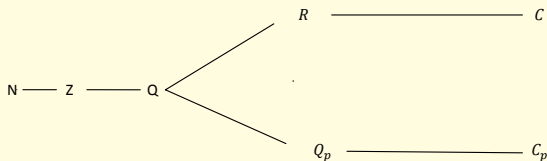
$$x = p^\nu \sum_{n=0}^{+\infty} x_n p^n, \quad \nu \in \mathbb{Z}, \quad x_n \in \{0, 1, \dots, p-1\}$$

$$-1 = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \dots + 2 \cdot 3^n + \dots \quad (p = 3)$$



2. p -Adic numbers, adeles and their functions

- Ostrowski theorem: Any nontrivial norm on \mathbb{Q} is equivalent to usual absolute value or to p -adic norm, where p is any prime number ($p = 2, 3, 5, 7, 11, \dots$).
- p -adic norm of $x \in \mathbb{Q}$: $|x|_p = |p^\nu \frac{a}{b}|_p = p^{-\nu}$, $\nu \in \mathbb{Z}$.
- Completion of \mathbb{Q} with respect to p -adic distance gives the field \mathbb{Q}_p of p -adic numbers, in analogous way to construction of the field \mathbb{R} of real numbers.
- \mathbb{Q} is dense in \mathbb{Q}_p and \mathbb{R} .
- \mathbb{Q}_p is totally disconnected space.



2. p -Adic numbers, adeles and their functions: some topological properties of p -adic numbers

- p -adic distances $d_p(x, y) = |x - y|_p$ are non-Archimedean (ultrametric): $d_p(x, y) \leq \max\{d_p(x, z), d_p(z, y)\}$
- open ball of radius r and centre a :

$$B_a(r^-) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}$$

- closed ball of radius r and centre a :

$$B_a(r) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}$$

- balls are simultaneously closed and open



All triangles are isosceles.



There is no partial intersection of balls.



Any point of a ball is its center.

2. p -Adic numbers, adeles and their functions

- There are mainly two kinds of analysis of p -adic variable:
 - (i) p -adic valued functions of p -adic variable
 - (ii) complex (real) valued functions of p -adic variable.
- Results of measurements are (rational) subset of real numbers – p -adic number cannot be directly measured
- Analysis of complex (real) valued functions of p -adic (and real) variables is necessary to connect models with measurements.

2. p -Adic numbers, adeles and their functions

- Real and p -adic numbers are unified by adeles. An **adele** α is an infinite sequence

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \dots, \alpha_p, \dots), \quad \alpha_\infty \in \mathbb{R}, \quad \alpha_p \in \mathbb{Q}_p$$

where for all but a finite set \mathcal{P} of primes p one has that $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$, i.e. p -adic integers.

- Space of adeles

$$\mathbb{A} = \bigcup_{\mathcal{P}} A(\mathcal{P}), \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p.$$



C. Chevalley (1909 - 1984)



A. Weil (1906 - 1998)

2. p -Adic numbers, adeles and their functions

Some connections of p -adic and real norms of rational numbers

$$|x|_{\infty}^c \times \prod_{p \in \mathbb{P}} |x|_p^c = 1, \text{ if } x \in \mathbb{Q}^{\times}$$

$$\chi_{\infty}(x) \times \prod_{p \in \mathbb{P}} \chi_p(x) = 1, \text{ if } x \in \mathbb{Q}$$

$$\chi_{\infty}(x) = \exp(-2\pi i x), \quad \chi_p(x) = \exp 2\pi i \{x\}_p$$

$$\prod_{p \in \mathbb{P}} \Omega(|x|_p) = \begin{cases} 1, & x \in \mathbb{Z}, \\ 0, & x \in \mathbb{Q} \setminus \mathbb{Z} \end{cases}$$

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \leq 1, \\ 0, & |x|_p > 1 \end{cases}$$

2. p -Adic numbers, adeles and their functions

- Numerical result of any measurement is a (real) rational number with some uncertainty (error): i.e. $x = \bar{x} \pm \Delta x$
- Numerical result of the corresponding theoretical model has to be equal to this rational number from the measurement.
- Note that p -adic number is not result of a measurement!
- Then the question: How p -adic numbers and p -adic analysis can be related to applications in physics and sciences?

2. p -Adic numbers, adeles and their functions

- Recall standard theoretical models

$$\textit{classical models} : \mathbb{R} \rightarrow \mathbb{R}$$

$$\textit{quantum models} : \mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{R}$$

- p -Adic applications

$$\textit{classical models}(\textit{genetic code}, \dots) : \mathbb{Q}_p \rightarrow \mathbb{R}$$

$$\textit{quantum models}(\textit{string theory}, \dots) : \mathbb{Q}_p \rightarrow \mathbb{C} \rightarrow \mathbb{R}$$

- p -Adic side of some phenomena emerges at a deeper level.

3. p -Adic strings

Volovich, Freund, Witten, Vladimirov, Arefeva, B. D., ...

String amplitudes (1987):

- standard crossing symmetric Veneziano amplitude

$$\begin{aligned} A_\infty(a, b) &= g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1} |1-x|_\infty^{b-1} d_\infty x \\ &= g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)} \end{aligned}$$

- p -adic crossing symmetric Veneziano amplitude

$$\begin{aligned} A_p(a, b) &= g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1} |1-x|_p^{b-1} d_p x \\ &= g_p^2 \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}} \end{aligned}$$

where $a = -s/2 - 1$ and $a, b, c \in \mathbb{C}$ and $a + b + c = 1$.

3. p -Adic strings

- p -Adic strings are strings which world-sheet is p -adic.
- Freund-Witten (adelic) product formula for ordinary and p -adic strings

$$A(a, b) = A_\infty(a, b) \prod_p A_p(a, b) = g_\infty^2 \prod_p g_p^2 = \text{const.}$$

- Ordinary and p -adic strings are at the equal footing
- Amplitude for real string $A_\infty(a, b)$, which is a special function, can be presented as product of inverse p -adic amplitudes, which are elementary functions.

3. p -Adic strings – Effective Lagrangian

≥ 1988 : Freund, Witten, Frampton, Okada, ..., ≥ 2000 :
Vladimirov, Volovich, Sen, Ghoshal, Zwiebach, Moeller,
Schnabl, Barnaby, Kamran, Minahan, Yang, Biswas, Arefeva,
Koshelev, Vernov, Joukovskaya, B.D., ...

- There is an effective field description of scalar open and closed p -adic strings. The corresponding Lagrangians are very simple and exact. They describe not only four-point scattering amplitudes but also all higher ones at the tree-level.
- The exact tree-level Lagrangian for effective scalar field φ which describes open p -adic string tachyon is

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \left[-\frac{1}{2} \varphi p^{-\frac{\square}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \right]$$

where $\square = -\partial_t^2 + \nabla^2$ is the D -dimensional d'Alembertian and metric with signature $(- + \dots +)$.

3. p -Adic strings

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \left[-\frac{1}{2} \varphi p^{-\frac{\square}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \right]$$

- Lagrangian with p -adic world-sheet B.D.

$$\mathcal{L}_p = \frac{m^D}{g^2} \frac{p^2}{p-1} \left[\frac{1}{2} \varphi \int_{\mathbb{R}} \left(\int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \chi_p(u) |u|_p^{\frac{k^2}{2m^2}} du \right) \tilde{\varphi}(k) \chi_{\infty}(kx) d^4 k \right. \\ \left. + \frac{1}{p+1} \varphi^{p+1} \right]$$

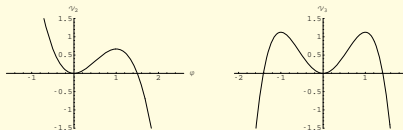
$$\int_{|x|_p > 1} \chi_p(u) |u|_p^s du = -p^s$$

- The corresponding potential $\mathcal{V}(\varphi)$ for this Lagrangian is $\mathcal{V}_p(\varphi) = -\mathcal{L}_p(\square = 0)$, which the explicit form is

$$\mathcal{V}_p(\varphi) = \frac{m^D}{g^2} \left[\frac{1}{2} \frac{p^2}{p-1} \varphi^2 - \frac{p^2}{p^2-1} \varphi^{p+1} \right].$$

It has local minimum $\mathcal{V}_p(0) = 0$. If $p \neq 2$ there are two local maxima at $\varphi = \pm 1$ and there is one local maximum $\varphi = +1$ when $p = 2$.

- The 2-adic string potential $\mathcal{V}_2(\varphi)$ (on the left) and 3-adic potential $\mathcal{V}_3(\varphi)$ (on the right) of the figure.



3. p -Adic strings – equation of motion

- The equation of motion is

$$p^{-\frac{\square}{2m_p^2}} \varphi = \varphi^p$$

- It has trivial solutions $\varphi = 0$ and $\varphi = 1$, and $\varphi = -1$ for $p \neq 2$.
- There are also solutions in any direction x^i and time t

$$\varphi(x^i) = p^{\frac{1}{2(p-1)}} \exp\left(-\frac{p-1}{2p \ln p} m^2 (x^i)^2\right)$$

$$\varphi(t) = p^{\frac{1}{2(p-1)}} \exp\left(\frac{p-1}{2p \ln p} m^2 t^2\right)$$

3. p -Adic strings

- There is also D -dimensional solution

$$\varphi(x) = p^{\frac{D}{2(p-1)}} \exp\left(-\frac{p-1}{2p \ln p} m^2 x^2\right), \quad x^2 = -t^2 + \sum_{i=1}^{D-1} x_i^2$$

- The above solutions can be obtained using identity

$$e^{A \partial_t^2} e^{B t^2} = \frac{1}{\sqrt{1-4AB}} e^{\frac{B t^2}{1-4AB}}, \quad 1-4AB > 0.$$

- p -Adic realization of tachyon condensation

3. p -Adic strings – Lagrangian for p -adic string sector

B.D.

Consider

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = m^D \sum_{n=1}^{+\infty} \frac{C_n}{g_n^2} \frac{n^2}{n-1} \left[-\frac{1}{2} \phi n^{-\frac{D}{2m^2}} \phi + \frac{1}{n+1} \phi^{n+1} \right]$$

where $\frac{C_n}{g_n^2} \frac{n^2}{n-1} = (-1)^{n-1}$. Take into account

$$\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s}) \zeta(s), \quad s = \sigma + i\tau, \quad \sigma > 0$$

which has analytic continuation to the entire complex s plane without singularities.

3. p -Adic strings – Lagrangian for p -adic string sector

- The corresponding Lagrangian is

$$L = m^D \left[-\frac{1}{2} \phi \left(1 - 2^{1-\frac{\square}{2m^2}} \right) \zeta \left(\frac{\square}{2m^2} \right) \phi + \phi - \frac{1}{2} \log(1 + \phi)^2 \right].$$

- The potential is

$$V(\phi) = -L(\square = 0) = m^D \left[\frac{1}{4} \phi^2 - \phi + \frac{1}{2} \log(1 + \phi)^2 \right],$$

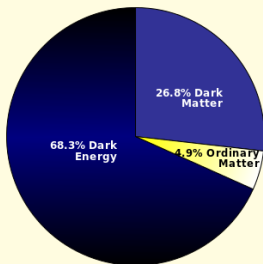
which has one local maximum $V(0) = 0$ and one local minimum at $\phi = 1$. It is singular at $\phi = -1$.

- The equation of motion is

$$\left(1 - 2^{1-\frac{\square}{2m^2}} \right) \zeta \left(\frac{\square}{2m^2} \right) \phi = \frac{\phi}{1 + \phi}, \quad \phi = 0, \quad \phi = 1.$$

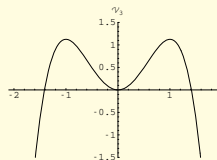
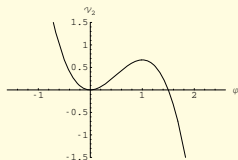
4. Possible gravitational and cosmological aspects

- If Einstein general theory of relativity is theory of gravity for the Universe as a whole, then there is only about 5% of ordinary matter, about 27% of dark matter and about 68% of dark energy.
- Conjecture: Dark matter and dark energy have p -adic origin.
- Modified gravity may be nonlocal.



4. Possible gravitational and cosmological aspects

- p -Adic inflation



5. Concluding remarks

- p -Adic numbers are not results of measurements, but can be useful in physical models.
- p -Adic strings are strings with world-sheet labeled by p -adic numbers.
- If there exist ordinary strings then there should exist also p -adic strings.
- p -Adic string theory is simpler than ordinary string theory, and seems to be useful for ordinary string theory.
- p -Adic strings may play significant role in description of dark side of the universe.
- Main applications in the near future in bioinformation systems.

5. Concluding remarks – some review references

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