

Closed string noncommutativity in the weakly curved background

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Outline of the talk

- 1 Open string noncommutativity
- 2 T-duality
- 3 Weakly curved background
- 4 Closed string noncommutativity
- 5 Concluding remarks

Open string noncommutativity

- Space-time looks different from the string point of view.
- Open string endpoints became noncommutative in the presence of the constant Kalb-Ramond field $B_{\mu\nu}$.
- Minimum action principle $\delta S = 0$

$$S(x) = \kappa \int_{\Sigma} d^2\xi \left(\frac{1}{2} \eta^{\alpha\beta} G_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu} \right) \partial_{\alpha} x^{\mu} \partial_{\beta} x^{\nu} ,$$

gives equations of motion and boundary conditions.

Open string noncommutativity

- Solution of the boundary conditions is of the form

$$x^\mu = q^\mu - \Theta^{\mu\nu} \int d\eta p_\nu(\eta),$$

where q^μ and p_ν are effective variables satisfying $\{q^\mu(\sigma), p_\nu(\bar{\sigma})\} = \delta^\mu{}_\nu \delta_s(\sigma, \bar{\sigma})$.

- Coordinate x^μ is the linear combination of the effective coordinate q^μ and effective momentum p_ν which produces the noncommutativity

$$\{x^\mu(0), x^\nu(0)\} = -2\Theta^{\mu\nu}, \quad \{x^\mu(\pi), x^\nu(\pi)\} = 2\Theta^{\mu\nu}.$$

Open string noncommutativity

- Effective action is of the form

$$S(q) = S(x)|_{bc} = \frac{1}{2} \int_{\Sigma} d^2\xi \eta^{\alpha\beta} G_{\mu\nu}^E \partial_{\alpha} q^{\mu} \partial_{\beta} q^{\nu},$$

where

$$G_{\mu\nu}^E = (G - 4BG^{-1}B)_{\mu\nu}, \quad \Theta^{\mu\nu} = -\frac{2}{\kappa} (G_E^{-1}BG^{-1})^{\mu\nu},$$

are effective metric and noncommutativity parameter, respectively.

Closed string T-duality

- T-duality connects physically equivalent theories with different backgrounds.
- There are two main consequences of the compactification on a circle:
 - momentum becomes quantized - $p = \frac{n}{R}$ ($n \in \mathbb{Z}$)
 - new states appear (winding modes)

$$x(2\pi) - x(0) = 2\pi RN.$$

- Mass squared of any state

$$M^2 = \frac{n^2}{R^2} + m^2 \frac{R^2}{\alpha'^2},$$

is invariant under exchanges $n \leftrightarrow m, R \leftrightarrow \alpha' / R$.

Closed string T-duality

- Compactification on a circle of radius R is equivalent to the compactification on a circle of radius *R .
- T-dual action *S is of the same form as the initial one but with background fields

$$^*G^{\mu\nu} \sim (G_E^{-1})^{\mu\nu}, \quad ^*B^{\mu\nu} \sim \Theta^{\mu\nu}.$$

Choice of the background fields

- $G_{\mu\nu} = \text{const.}$ and $B_{\mu\nu} = b_{\mu\nu} + h_{\mu\nu}(x) \equiv b_{\mu\nu} + \frac{1}{3}H_{\mu\nu\rho}x^\rho$.
- $b_{\mu\nu}$ and $H_{\mu\nu\rho}$ are constants and $H_{\mu\nu\rho}$ is infinitesimal.
- Background fields satisfy the space-time equations of motion (consistency conditions). Ricci tensor $R_{\mu\nu}$ is proportional to $H_{\mu\nu}^2$, so, in **linear** approximation in $H_{\mu\nu\rho}$ space-time metric can be considered as constant.

Generalized Buscher rules

- There are two steps in generalized Buscher procedure:
 - localization of global shift symmetry $\delta x^\mu = \lambda^\mu$

$$\partial_\alpha x^\mu \rightarrow D_\alpha x^\mu = \partial_\alpha x^\mu + v_\alpha^\mu,$$

- $x^\mu \rightarrow \Delta x_{inv}^\mu = \int_P d\xi^\alpha D_\alpha x^\mu$. (new step)

T-dual action

- $x^\mu \rightarrow V^\mu = -\kappa \Theta_0^{\mu\nu} y_\nu + (g_E^{-1})^{\mu\nu} \tilde{y}_\nu$.
- ${}^*G^{\mu\nu} = (G_E^{-1})^{\mu\nu}(\Delta V)$ and ${}^*B^{\mu\nu} = \frac{\kappa}{2} \Theta^{\mu\nu}(\Delta V)$.
- The T-dual action is of the form

$${}^*S = \frac{\kappa^2}{2} \int_{\Sigma} d^2\xi \partial_+ y_\mu \Theta_-^{\mu\nu}(\Delta V) \partial_- y_\nu,$$

where

$$\Theta_{\pm}^{\mu\nu}(x) = -\frac{2}{\kappa} (G_E^{-1}(x) \Pi_{\pm}(x) G^{-1})^{\mu\nu}, \quad \Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}.$$

T-dual transformations in the canonical form

$$x'^{\mu} = \frac{1}{\kappa} \star \pi^{\mu} - \kappa \Theta_0^{\mu\nu} \beta_{\nu}^0(V) - (g_E^{-1})^{\mu\nu} \beta_{\nu}^1(V),$$

$$y'_{\mu} = \frac{1}{\kappa} \pi_{\mu} - \beta_{\mu}^0(x).$$

Infinitesimally small terms (contain β_{μ}^{α}) are corrections with respect to the constant background case. These terms are the source of noncommutativity.

Calculation of Poisson brackets

$$\Delta X_\mu(\sigma, \sigma_0) = \int_{\sigma_0}^{\sigma} d\sigma_1 X'_\mu(\sigma_1), \quad \Delta Y_\mu(\sigma, \sigma_0) = \int_{\sigma_0}^{\sigma} d\sigma_1 Y'_\mu(\sigma_1).$$

From

$$\{X'_\mu(\sigma), Y'_\nu(\bar{\sigma})\} = K'_{\mu\nu}(\sigma)\delta(\sigma - \bar{\sigma}) + L_{\mu\nu}(\sigma)\delta'(\sigma - \bar{\sigma}),$$

we obtain

$$\{X_\mu(\sigma), Y_\nu(\bar{\sigma})\} = -[K_{\mu\nu}(\sigma) - K_{\mu\nu}(\bar{\sigma}) + L_{\mu\nu}(\bar{\sigma})]\theta(\sigma - \bar{\sigma}).$$

Taking $\sigma \rightarrow 2\pi + \sigma$ and $\bar{\sigma} \rightarrow \sigma$ we get

$$\{X_\mu(2\pi + \sigma), Y_\nu(\sigma)\} = -[K_{\mu\nu}(2\pi + \sigma) - K_{\mu\nu}(\sigma) + L_{\mu\nu}(\sigma)].$$

Origin of noncommutativity

Calculation of Poisson brackets is in fact a calculation of Poisson brackets of the coordinate sigma derivatives. Sigma derivatives of the coordinates are given by T-dual transformations in the canonical form. Since the T-dual transformation of the coordinate is given as linear combination of the coordinates and their canonically conjugated momenta from the T-dual space, we get the noncommutativity of the coordinates. For constant background case there is no noncommutativity

$$\pi_\mu = \kappa y'_\mu, \quad \star \pi^\mu = \kappa X'^\mu,$$

$$\{\pi_\mu, \pi_\nu\} = 0 \Rightarrow \{y_\mu, y_\nu\} = 0.$$

Definitions

- $*B^{\mu\nu}$ depends on $\Delta V(y, \tilde{y})$

$$*B^{\mu\nu} = b^{\mu\nu} + Q^{\mu\nu}{}_{\rho} \Delta V^{\rho} .$$

- Christofel symbol for $G_{\mu\nu}^E$ is denoted by $\Gamma_{\mu,\nu\rho}^E$.
- $\tilde{y}'_{\mu} = \dot{y}_{\mu}$.

Results

$$\begin{aligned}
 \{y_\mu(2\pi + \sigma), y_\nu(\sigma)\} &= -\frac{2\pi}{\kappa} H_{\mu\nu\rho} N^\rho, \\
 \{y_\mu(2\pi + \sigma), \tilde{y}_\nu(\sigma)\} + \{y_\mu(\sigma), \tilde{y}_\nu(2\pi + \sigma)\} \\
 &= -\frac{4\pi}{\kappa^2} H_{\mu\nu\rho} p^\rho + \frac{\pi}{\kappa} \left(3\Gamma_{\rho,\mu\nu}^E - 8H_{\mu\nu\lambda} b^\lambda{}_\rho \right) N^\rho, \\
 \{\tilde{y}_\mu(2\pi + \sigma), \tilde{y}_\nu(\sigma)\} &= \frac{2\pi}{\kappa} \left[-H_{\mu\nu\rho} - 6g_{\mu\alpha} Q^{\alpha\beta}{}_\rho g_{\beta\nu} \right] N^\rho \\
 + \frac{2\pi}{\kappa} \left[2H_{\mu\nu}{}^\lambda g_{\lambda\rho} + 3 \left(\Gamma_{\mu,\nu\lambda}^E - \Gamma_{\nu,\mu\lambda}^E \right) b^\lambda{}_\rho \right] N^\rho \\
 + \frac{\pi}{\kappa^2} \left[3 \left(\Gamma_{\mu,\nu\rho}^E - \Gamma_{\nu,\mu\rho}^E \right) - 8H_{\mu\nu\lambda} b^\lambda{}_\rho \right] p^\rho.
 \end{aligned}$$

Winding numbers and momenta

$$N^\mu = \frac{1}{2\pi} [x^\mu(2\pi + \sigma) - x^\mu(\sigma)] , \quad {}^*N_\mu = \frac{1}{2\pi} [y_\mu(2\pi + \sigma) - y_\mu(\sigma)] ,$$

$$p_\mu = \frac{1}{2\pi} \int_\sigma^{2\pi+\sigma} d\eta \pi_\mu(\eta) , \quad {}^*p^\mu = \frac{1}{2\pi} \int_\sigma^{2\pi+\sigma} d\eta {}^*\pi^\mu(\eta) .$$

Algebra of winding numbers and momenta

For calculation we need expressions calculated earlier and

$$\Delta y_\mu(2\pi, 0) = 2\pi^* N_\mu, \quad \Delta \tilde{y}_\mu(2\pi, 0) = 2\pi^* P_\mu,$$

$$\Delta x^\mu = 2\pi N^\mu, \quad \Delta \tilde{x}^\mu = 2\pi P^\mu.$$

Algebra of winding numbers and momenta

$$\{^*N_\mu, ^*N_\nu\} = \frac{1}{\pi\kappa} H_{\mu\nu\rho} N^\rho,$$

$$\{^*N_\mu, ^*P_\nu\} = \frac{1}{\pi\kappa} H_{\mu\nu\rho} P^\rho - \frac{3}{4\pi\kappa} \Gamma_{\rho,\mu\nu}^E N^\rho,$$

$$\begin{aligned} \{^*P_\mu, ^*P_\nu\} &= -\frac{1}{\pi\kappa} \left(H_{\mu\nu\rho} - 6g_{\mu\alpha} Q^{\alpha\beta}{}_{\rho} g_{\beta\nu} \right) N^\rho \\ &+ \frac{1}{\pi} \left[-\frac{3}{2\kappa} (\Gamma_{\mu,\nu\rho}^E - \Gamma_{\nu,\mu\rho}^E) + \frac{4}{\kappa} H_{\mu\nu\sigma} (G^{-1}b)^\sigma{}_\rho \right] P^\rho. \end{aligned}$$

Concluding remarks

- If there is a nontrivial winding in the weakly curved background, using T-duality transformation laws we get closed string noncommutativity.
- Poisson brackets are, in general, proportional to the linear combination of the T-dual winding number and momenta modes.
- We obtain the algebra of the T-dual winding number and momenta.