

**8th MATHEMATICAL PHYSICS MEETING:
Summer School and Conference on
Modern Mathematical Physics**

$M \cap \Phi$

**Conformal invariance for scalar
and Dirac particles in
Riemannian spacetimes: new
results**

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OUTLINE

- **Covariant Klein-Gordon equation and conformal invariance for massless particles.**
Fifty-year history
- **Conformal invariance and new (Hermitian) form of the Klein-Gordon equation. Conformal symmetry for a pointlike scalar particle (Higgs boson)**
- **Conformal symmetries of Hamiltonians**
- **Exact Foldy-Wouthyusen transformation**
- **Inclusion of electromagnetic interactions**
- **Comparison of scalar and Dirac particles**
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**Covariant Klein-Gordon equation
and conformal invariance for
massless particles. *Fifty-year
history***

Conformal invariance for a massless particle



R. Penrose



**N.A. Chernikov
(1928 – 2007)**



E. Tagirov

R. Penrose, In: Relativity, Groups and Topology.

London: Gordon and Breach, **1964**, p. 565.

N. Chernikov and E. Tagirov, Ann. Inst. Henri Poincarè **9**, 109 (1968).

Covariant Klein-Gordon equation with the nonminimal coupling:

$$\left(\frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} + m^2 - \lambda R \right) \psi = 0, \quad \lambda = \frac{1}{6}.$$

Non-minimal coupling with the scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^{\lambda}_{\mu\lambda\nu}.$$

*The sign of the Penrose-Chernikov-Tagirov term depends
on the definition of R*

Republication of: Conformal treatment of infinity

Roger Penrose

The utility of this idea rests on the fact that the zero rest-mass free-field equations for each spin value are conformally invariant if interpreted suitably. For example, for spin zero, if the wave equation is written as

$$\left\{ \nabla_{\mu} \nabla^{\mu} + \frac{R}{6} \right\} \phi = 0$$

where R is the scalar curvature and ∇_{μ} denotes covariant derivative—both according to the metric $g_{\mu\nu}$ of \mathcal{M} , then

$$\left\{ \tilde{\nabla}_{\mu} \tilde{\nabla}^{\mu} + \frac{\tilde{R}}{6} \right\} \tilde{\phi} = 0$$

where $\tilde{\nabla}_{\mu}$, \tilde{R} refer to the metric $\tilde{g}_{\mu\nu} = \Omega^{-2}g_{\mu\nu}$ of $\tilde{\mathcal{M}}$ and where
 $\tilde{\phi} = \Omega\phi$.

N. Chernikov and E. Tagirov, Ann. Inst. Henri Poincaré **9**, 109 (1968)

$$\square - \frac{n-2}{4(n-1)} R = \Omega^{\frac{n+2}{2}} \left(\square' - \frac{n-2}{4(n-1)} R' \right) \Omega^{\frac{2-n}{2}},$$

$$\psi' = \Omega^{\frac{n-2}{2}} \psi, \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}.$$

n is a number of dimensions

But the only discovered pointlike scalar particle (Higgs boson) is massive!

Chernikov and Tagirov discussed an importance of non-minimal coupling with the scalar curvature for massive particles

New step ahead (Hamiltonian approach):

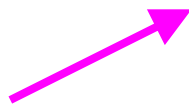
A. Accioly and H. Blas, Exact Foldy-Wouthuysen transformation for real spin-0 particle in curved space, PHYSICAL REVIEW D **66**, 067501 (2002).

Static metric in isotropic coordinates:

$$ds^2 = V(\mathbf{x})^2 (dx^0)^2 - W(\mathbf{x})^2 (d\mathbf{x})^2.$$

Feshbach-Villars transformation (useless for massless particles):

$$\psi = \phi + \chi, \quad \frac{i}{m} \frac{\partial \psi}{\partial t} = \phi - \chi.$$



the mass in the denominator!

$$\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{V^2}{2W^3} \Delta W + \frac{V}{2W^2} \Delta V - \frac{1}{6} V^2 R},$$

$$-\frac{1}{6} V^2 R = \frac{1}{6} F \Delta F - \frac{V^2}{2W^3} \Delta W - \frac{V}{2W^2} \Delta V,$$

$$\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{1}{6} F \Delta F}, \quad F = \frac{V}{W}.$$

$$\text{If } m = 0, \quad \mathcal{H}_{FW} \left((g')^{\mu\nu} = \frac{g^{\mu\nu}}{\Omega(\mathbf{x})^2} \right) = \mathcal{H}_{FW} (g^{\mu\nu}).$$

$$ds^2 = V^2 (dx^0)^2 - W^2 (d\mathbf{x})^2.$$

All terms except for the first term are conformally invariant
**Conformal transformation changes only such terms in the Foldy-
 Wouthuysen Hamiltonian which are proportional to the particle mass**

**But it is only a shadow of the conformal invariance,
 because $m \neq 0$!**

Hamiltonian approach in classical general relativity:

$$\mathcal{H}_{class} = \sqrt{\frac{m^2 - G^{ij} p_i p_j}{g^{00}}} + \frac{g^{0i} p_i}{g^{00}}, \quad G^{ij} = g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}}.$$

G. Cognola, L. Vanzo, and S. Zerbini, Gen. Relativ. Gravit. **18**, 971 (1986).

$$\text{If } m = 0, \quad \mathcal{H}_{class} \left((g')^{\mu\nu} = \frac{g^{\mu\nu}}{\Omega(\mathbf{x})^2} \right) = \mathcal{H}_{class} (g^{\mu\nu}).$$

The second-order form of this classical Hamiltonian equation is

$$g^{\mu\nu} p_\mu p_\nu - m^2 + \cancel{\lambda R} = 0, \quad \lambda = 0!$$


The classical equations contrary to quantum mechanical ones correspond to the minimal coupling

Scalar particle in general inertial and gravitational fields and conformal invariance revisited

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(Received 29 May 2013; published 5 August 2013)

- 1. Exact Foldy-Wouthuysen transformation for**
 - i) general static metric**
 - ii) frame rotating in the Kerr field approximated by a spatially isotropic metric**
- 2. Foldy-Wouthuysen Hamiltonian is conformally invariant for massless particles and conformally symmetric for massive ones**
- 3. Proof of similarity of conformal transformations for scalar and Dirac particles**



**Conformal invariance and new
(Hermitian) form of the Klein-
Gordon equation. Conformal
symmetry for a pointlike scalar
particle
(Higgs boson)**

$$\left(\frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} + m^2 - \lambda R \right) \psi = 0.$$

$$\psi' = \Omega^{\frac{n-2}{2}} \psi, \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}.$$

The nonunitary transformation

$$\Phi = \sqrt{g^{00}} \sqrt{-g} \psi, \quad g = \det g_{\mu\nu}, \quad g' = \Omega^{2n} g.$$

Φ is *invariant* relative to conformal transformations

We multiply the KFG equation from left by $\sqrt{\frac{\sqrt{-g}}{g^{00}}}$.

Hermitian form of the KFG equation:

$$\left(\frac{1}{\sqrt{g^{00}} \sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \frac{1}{\sqrt{g^{00}} \sqrt{-g}} - \frac{\lambda R}{g^{00}} + \frac{m^2}{g^{00}} \right) \Phi = 0.$$

For a massless particle

$$\left(\frac{1}{\sqrt{g^{00}} \sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \frac{1}{\sqrt{g^{00}} \sqrt{-g}} - \frac{\lambda R}{g^{00}} \right)$$
$$= \left(\frac{1}{\sqrt{g^{00}} \sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \frac{1}{\sqrt{g^{00}} \sqrt{-g}} - \frac{\lambda R}{g^{00}} \right) .$$

This part of the KG equation can be transformed:

Denotations:

$$f = \sqrt{g^{00}} \sqrt{-g}, \quad \Gamma^i = \sqrt{-g} g^{0i},$$

$$G^{ij} = g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}}, \quad \Upsilon = \frac{1}{2f} \left\{ \partial_i, \Gamma^i \right\} \frac{1}{f} = \frac{1}{2} \left\{ \partial_i, \frac{g^{0i}}{g^{00}} \right\},$$

$$\Lambda = -\frac{f_{,0,0}}{f} - \left(\frac{g^{0i}}{g^{00}} \right)_{,i} \frac{f_{,0}}{f} - 2 \frac{g^{0i}}{g^{00}} \frac{f_{,0,i}}{f} - \left(\frac{g^{0i}}{g^{00}} \right)_{,0} \frac{f_{,i}}{f}$$

$$- \frac{1}{2} \left(\frac{g^{0i}}{g^{00}} \right)_{,0,i} - \frac{1}{2f^2} \left(\frac{g^{0i}}{g^{00}} \right)_{,i} \Gamma^j_{,j} - \frac{g^{0i}}{2f^2 g^{00}} \Gamma^j_{,i,j}$$

$$+ \frac{1}{4f^4} \left(\Gamma^i_{,i} \right)^2 - \left(\frac{G^{ij}}{g^{00}} \right)_{,i} \frac{f_{,j}}{f} - \frac{G^{ij}}{g^{00}} \frac{f_{,i,j}}{f} - \frac{\lambda R}{g^{00}}.$$

Equivalent Hermitian form of the KG equation:

$$\left[(\partial_0 + \Upsilon)^2 + \partial_i \frac{G^{ij}}{g^{00}} \partial_j + \Lambda + \frac{m^2}{g^{00}} \right] \Phi = 0.$$

Υ , G^{ij} / g^{00} , and Λ are invariant relative to conformal transformations

We can extend conformal symmetry on massive particles. The conformal-like transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1} m$$

conserves the operator [...] acting on Φ and, therefore, conserves Φ when $\lambda=1/6$ (generally, $(n-2)/[4(n-1)]$). As a result, the operator acting on Φ in the equation

$$\left(\frac{1}{\sqrt{g^{00}} \sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \frac{1}{\sqrt{g^{00}} \sqrt{-g}} - \frac{\lambda R}{g^{00}} + \frac{m^2}{g^{00}} \right) \Phi = 0$$

remains unchanged. The initial covariant Klein-Gordon equation

$$\left(\frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} + m^2 - \frac{n-2}{4(n-1)} R \right) \psi = 0$$

has following properties:

$$\square + m^2 - \frac{n-2}{4(n-1)} R$$

$$= \Omega^{\frac{n+2}{2}} \left(\square' + m'^2 - \frac{n-2}{4(n-1)} R' \right) \Omega^{\frac{2-n}{2}},$$

$$\psi' = \Omega^{\frac{n-2}{2}} \psi \quad \text{when} \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1} m.$$

The conformal symmetry has extended on a pointlike scalar particle (Higgs boson)!



Conformal symmetries of Hamiltonians

Feshbach-Villars transformation to a Hamiltonian form for a *massive* particle

$$\left[(\partial_0 + \Upsilon)^2 + \mathcal{T} \right] \Phi = 0, \quad \mathcal{T} = \partial_i \frac{G^{ij}}{g^{00}} \partial_j + \Lambda + \frac{m^2}{g^{00}}$$

$$\Phi = \phi + \chi, \quad i(\partial_0 + \Upsilon)\Phi = m(\phi - \chi).$$

***m* appears in a denominator!**

Generalized Feshbach-Villars transformation for both massive and massless particles

The wave function in the Feshbach-Villars representation is given by

$$\Psi = \frac{1}{2} \begin{pmatrix} \Phi + \frac{i}{m} (\partial_0 + \Upsilon) \Phi \\ \Phi - \frac{i}{m} (\partial_0 + \Upsilon) \Phi \end{pmatrix}.$$

It was proved that a similar transformation can be performed with the use of any nonzero parameter instead of the particle mass, ***m***

Successive generalized Feshbach-Villars and Foldy-Wouthyusen transformations

The method has been developed in

A.J. Silenko, Hamilton operator and the semiclassical limit for scalar particles in an electromagnetic field, Theor. Math. Phys. **156**, 1308 (2008).

$$\psi = \phi + \chi, \quad \frac{i}{N}(\partial_0 + \Upsilon)\psi = \phi - \chi. \quad N \text{ is an arbitrary nonzero real parameter}$$

$$i(\partial_0 + \Upsilon)\Psi = \left[\frac{N}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \frac{T}{2N} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right] \Psi, \quad \Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

The Pauli matrices ρ_i can be used

$$i(\partial_0 + \Upsilon)\Psi = \frac{1}{2N} \left[\rho_3 (N^2 + T) + i\rho_2 (-N^2 + T) \right] \Psi.$$

Therefore, we obtain the following generalized Feshbach-Villars Hamiltonian:

$$\mathcal{H}_{gFV} = \rho_3 \frac{N^2 + T}{2N} + i\rho_2 \frac{-N^2 + T}{2N} - i\Upsilon.$$

This Hamiltonian *is not changed* by the conformal-like transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1} m.$$

General method of the Foldy-Wouthyusen transformation

$$\mathcal{H}_{gFV} = \rho_3 \mathcal{M} + \mathcal{E} + \mathcal{O}, \quad \rho_3 \mathcal{M} = \mathcal{M} \rho_3, \quad \rho_3 \mathcal{E} = \mathcal{E} \rho_3, \quad \rho_3 \mathcal{O} = -\mathcal{O} \rho_3,$$

$$\mathcal{M} = \frac{N^2 + T}{2N}, \quad \mathcal{E} = -i\Upsilon, \quad \mathcal{O} = i\rho_2 \frac{-N^2 + T}{2N},$$

$$U = \frac{\mathcal{E} + \mathcal{M} + \rho_3 \mathcal{O}}{\sqrt{2\mathcal{E}(\mathcal{E} + \mathcal{M})}}, \quad \varepsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2}, \quad \mathcal{F} = \mathcal{E} - i \frac{\partial}{\partial t}.$$

In the case under consideration,

$$U = \frac{\varepsilon + N + \rho_1(\varepsilon - N)}{2\sqrt{\varepsilon N}}, \quad \varepsilon = \sqrt{T}, \quad \mathcal{H}' = \rho_3 \varepsilon + \mathcal{E}' + \mathcal{O}',$$

$$\mathcal{E}' = \mathcal{E} + \frac{1}{2\sqrt{\varepsilon}} \left[\sqrt{\varepsilon}, \left[\sqrt{\varepsilon}, \mathcal{F} \right] \right] \frac{1}{\sqrt{\varepsilon}}, \quad \mathcal{O}' = \rho_1 \frac{1}{2\sqrt{\varepsilon}} \left[\varepsilon, \mathcal{F} \right] \frac{1}{\sqrt{\varepsilon}}.$$

The transformed (intermediate) Hamiltonian describing the both massive and massless particles does not contain N and is not changed by the conformal-like transformation!

Final approximate Foldy-Wouthuysen Hamiltonian

$$\mathcal{H}_{FW} = \rho_3 \varepsilon + \mathcal{E}', \quad \varepsilon = \sqrt{T}.$$

The conformal-like transformation does not change the Foldy-Wouthuysen Hamiltonian!

Conformal symmetries of the generalized Feshbach-Villars and Foldy-Wouthuysen Hamiltonians have been proved
in the general form!



Exact Foldy-Wouthyusen transformation

Sufficient condition of exact Foldy-Wouthuysen transformation

$$[\mathcal{M}, \mathcal{O}] = [\mathcal{F}, \mathcal{O}] = 0 \quad \Rightarrow \quad \mathcal{H}_{FW} = \rho_3 \sqrt{T} - i\Upsilon.$$

1. Foldy-Wouthuysen transformation is exact for *any static metric*

In many important cases, the spacetime metric can be represented in a static form with an appropriate coordinate transformation. For example, it can be made for de Sitter and anti-de Sitter spaces)

Static metric in isotropic coordinates

Exact Foldy-Wouthuysen transformation has been performed by Accioly and H. Blas (2002) for massive particles

$$ds^2 = V(\mathbf{x})^2 (dx^0)^2 - W(\mathbf{x})^2 (d\mathbf{x})^2.$$

The obtained result formally coincides with that by
Accioly and Blas

$$\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{V^2}{2W^3} \Delta W + \frac{V}{2W^2} \Delta V - \frac{1}{6} V^2 R,}$$

$$-\frac{1}{6} V^2 R = \frac{1}{6} F \Delta F - \frac{V^2}{2W^3} \Delta W - \frac{V}{2W^2} \Delta V,$$

$$\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{1}{6} F \Delta F}, \quad F = \frac{V}{W}.$$

But we already have a right to consider the case of $m=0$
showing the conformal invariance!

$$\text{If } m=0, \quad \mathcal{H}_{FW} \left((g')^{\mu\nu} = \frac{g^{\mu\nu}}{\Omega(\mathbf{x})^2} \right) = \mathcal{H}_{FW} (g^{\mu\nu}).$$

2. Frame rotating in the Kerr field approximated by a spatially isotropic metric

All effects of the Schwarzschild gravitational field and the frame rotation are **exactly** described in *isotropic* Arnowitt-Deser-Misner coordinates

Effects of rotation of the *Kerr* source are described in these coordinates within terms of order of $O(a^2 r^{-4})$

total mass M , total angular momentum $J=Mca$

This case covers an observer on the ground of the Earth or on a satellite. It reproduces not only the well-known effects of the rotating frame but also the Lense-Thirring effect.

Frame rotating in the Kerr field: An approximation by a spatially isotropic metric

$$d\mathbf{x}' = d\mathbf{x} - [\mathbf{\Omega}(r) \times \mathbf{x}] dx^0, \quad \mathbf{\Omega}(r) = \boldsymbol{\omega}(r) - \mathbf{o}, \quad \mathbf{o} = \text{const},$$

$$ds^2 = V^2(r) (dx^0)^2 - W^2(r) (d\mathbf{x} - \mathbf{K} dx^0) (d\mathbf{x} - \mathbf{K} dx^0),$$

$$\mathbf{K} = \mathbf{\Omega} \times \mathbf{x},$$

$$g^{00} = \frac{1}{V^2(r)}, \quad g^{0i} = \frac{K^i}{V^2(r)}, \quad G^{ij} = -\frac{\delta^{ij}}{W^2(r)}.$$

In isotropic spherical coordinates ($\mathbf{\Omega} = \Omega \mathbf{e}_z$),

$$ds^2 = \left[V^2(r) - W^2(r) \Omega^2(r) r^2 \sin^2 \theta \right] (dx^0)^2$$

$$- W^2(r) \left[dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2 + 2\Omega(r) \sin^2 \theta dx^0 d\phi) \right].$$

Exact Foldy-Wouthyusen Hamiltonian

$$T = m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{V}{2W^2} \left[F \left(\frac{2W'_r}{r} + W''_{rr} \right) + \frac{2V'_r}{r} + V''_{rr} \right] - \frac{1}{6} V^2 R,$$

$$-\frac{1}{6} V^2 R = \frac{1}{6} F \Delta F - \frac{V}{2W^2} \left[F \left(\frac{2W'_r}{r} + W''_{rr} \right) + \frac{2V'_r}{r} + V''_{rr} \right] + \frac{1}{12} \Omega_r'^2 r^2 \sin^2 \theta,$$


$$\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{1}{6} F \Delta F + \frac{1}{12} \Omega_r'^2 r^2 \sin^2 \theta + \mathbf{\Omega} \cdot \mathbf{l}},$$

$$\mathbf{\Omega} = -\mathbf{0} \quad \text{for rotating frame,} \quad \mathbf{\Omega} = \frac{2G\mathbf{J}}{r^3} \quad \text{for Lense-Thirring metric,}$$

$$\mathbf{\Omega} = \frac{2G\mathbf{J}}{r^3} \left[1 - \frac{3GM}{r} + \frac{21G^2 M^2}{4r^2} + \mathcal{O}\left(\frac{a^2}{r^2}\right) \right] \quad \text{for approximate Kerr metric,}$$

$\mathbf{l} = \mathbf{r} \times \mathbf{p}$ is operator of angular momentum.

The Hamiltonian *is not changed* by the conformal-like transformation



Inclusion of electromagnetic interactions

**Electromagnetic interactions can be added
as follows:**

$$\left[g^{\mu\nu} (\nabla_{\mu} + ieA_{\mu}) (\nabla_{\nu} + ieA_{\nu}) + m^2 - \lambda R \right] \psi = 0 .$$

Simple derivation leads to

$$\left[\frac{1}{\sqrt{-g}} (\partial_{\mu} + ieA_{\mu}) \sqrt{-g} g^{\mu\nu} (\partial_{\nu} + ieA_{\nu}) + m^2 - \lambda R \right] \psi = 0 .$$

**Equivalent (relative to conformal-like
transformations) form of this equation is given by**

$$\left[(D_0 + \Upsilon')^2 + D_i \frac{G^{ij}}{g^{00}} D_j + \Lambda + \frac{m^2}{g^{00}} \right] \Phi = 0,$$

Λ is the same

$$\Upsilon' = \frac{1}{2} \left\{ D_i, \frac{g^{0i}}{g^{00}} \right\}, \quad D_{\mu} = \partial_{\mu} + ieA_{\mu}.$$

Generalized Feshbach-Villars transformation for both massive and massless particles

$$\left[(\partial_0 + \Upsilon')^2 + T' \right] \Phi = 0, \quad T' = D_i \frac{G^{ij}}{g^{00}} D_j + \Lambda + \frac{m^2}{g^{00}},$$

$$\Phi = \phi + \chi, \quad i(D_0 + \Upsilon')\Phi = N(\phi - \chi).$$

The nonunitary transformation results in the following generalized Feshbach-Villars Hamiltonian:

$$\mathcal{H}_{g_{FV}} = \rho_3 \frac{N^2 + T'}{2N} + i\rho_2 \frac{-N^2 + T'}{2N} - i\Upsilon'.$$

This Hamiltonian *is not changed* by the conformal-like transformation $g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $m' = \Omega^{-1} m$.

The considered equations *do not change* their conformal properties when electromagnetic interactions are included



Comparison of scalar and Dirac particles



**Analysis of properties of conformal transformations
for Dirac particles has been fulfilled in**

PHYSICAL REVIEW D **88**, 045004 (2013)

**Scalar particle in general inertial and gravitational fields
and conformal invariance revisited**

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Conformal transformations for Dirac particles

gravity + electromagnetism

Covariant Dirac equation:

$$\left(i\hbar\gamma^a D_a - mc\right)\psi = 0, \quad D_a = e_a^\mu \partial_\mu + \frac{i}{4}\sigma^{bc}\Gamma_{bca}.$$

General nonunitary transformation brings this equation to the Hermitian Hamiltonian form:

$$\Phi = \sqrt{\sqrt{-g}e_{\hat{0}}^0} \psi.$$

Yu.N. Obukhov, A.J. Silenko, and O.V. Teryaev, Phys. Rev. D **84**, 024025 (2011).

The Hamiltonian *is not changed* by the conformal-like transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1}m.$$

Dirac and Foldy-Wouthuysen Hamiltonians are conformally invariant when $m=0$!

Wave function of the initial covariant Dirac equation possesses the following conformal property:

$$\psi' = \Omega^{3/2}\psi \quad \text{when} \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1}m.$$

Squared Dirac equation:

$$\left[\pi^i \pi_i - \frac{\hbar}{2} \sigma^{\alpha\beta} \left(\frac{q}{c} F_{\alpha\beta} + m \Phi_{\alpha\beta} \right) + \frac{\hbar^2}{4} R \right. \\ \left. + \frac{\hbar^2}{16} (2\Gamma^i_{\alpha\beta} \Gamma_i^{\alpha\beta} + i\varepsilon^{\alpha\beta\mu\nu} \Gamma^i_{\alpha\beta} \Gamma_{i\mu\nu} \gamma_5) - m^2 c^2 \right] \psi = 0,$$

← electromagnetic field tensor

For the squared Dirac equation, $\lambda=1/4!$

Summary

- The covariant Klein-Gordon equation is presented in a new (Hermitian) form and conformal symmetry for a massive pointlike scalar particle (**Higgs boson**) is found. Conformal transformations for a massless particle and conformal-like transformations for a massive one do not change the form of the obtained equation
- Generalized Feshbach-Villars transformation and Foldy-Wouthyusen are performed for both massive and massless scalar particles in arbitrary gravitational fields. Conformal symmetries of relativistic Hamiltonians are found in the general case.
- Exact Foldy-Wouthyusen transformations are fulfilled for an arbitrary static metric and for a frame rotating in the Kerr field approximated by a spatially isotropic metric
- It is proven that inclusion of electromagnetic interactions does not change conformal properties of the considered equations
- Conformal symmetries of equations for scalar and Dirac particles are very similar



Thank you for your attention