Killing-Yano tensors on toric Sasaki-Einstein spaces

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Killing forms (1)

A **conformal Killing-Yano tensor** (also called **conformal Yano tensor** or **conformal Killing form** or **twistor form**) of rank p on a (pseudo-) Riemannian manifold (M, g) is a p -form ω which satisfies [Yano, 1952]

$$\nabla_X \omega = \frac{1}{p+1} X \lrcorner \, d\omega - \frac{1}{n-p+1} X^* \wedge d^* \omega,$$

for any vector field X on M, where n is the dimension of M , X^* is the 1-form dual to the vector field X with respect to the metric g , $\lrcorner$ is the operator dual to the wedge product and d^* is the adjoint of the exterior derivative d.

Killing forms (2)

If ω is co-closed then we obtain the definition of a **Killing-Yano tensor**, also called **Yano tensor** or **Killing form** :

$$\omega_{i_1 \dots i_{k-1}(i_k;j)} = 0\,.$$

Here a semicolon precedes an index of covariant differentiation associated with the Levi-Civita connection and a round bracket denotes a symmetrization over the indices within.
Moreover, a Killing form ω is said to be a **special Killing form** if it satisfies for some constant c the additional equation

$$\nabla_X(d\omega) = cX^* \wedge \omega\,,$$

for any vector field X on M .

Killing forms (3)

There is also a symmetric generalization of the Killing vectors:

A symmetric tensor $K_{i_1 \cdots i_r}$ of rank $r > 1$ satisfying the generalized Killing equation

$$K_{(i_1 \cdots i_r ; j)} = 0,$$

is called a **Stäckel-Killing tensor** .

Killing forms (4)

The conserved quantities associated with Killing tensors are given by the following proposition:

For any geodesic γ with tangent vector $\dot{\gamma}^i$

$$Q_K = K_{i_1 \dots i_r} \dot{\gamma}^{i_1} \cdots \dot{\gamma}^{i_r},$$

is constant along γ .

Let us note that there is an important connection between these two generalizations of the Killing vectors. To wit, given two Killing-Yano tensors $\psi^{i_1, \dots, i_k}$ and $\sigma^{i_1, \dots, i_k}$ there is a Stäckel-Killing tensor of rank 2 :

$$K_{ij}^{(\psi,\sigma)} = \psi_{i i_2 \dots i_k} \sigma_j{}^{i_2 \dots i_k} + \sigma_{i i_2 \dots i_k} \psi_j{}^{i_2 \dots i_k}.$$

This fact offers a method to generate higher order integrals of motion by identifying the complete set of Killing-Yano tensors.

Sasakian geometry (1)

Contact structures(1)

A $(2n+1)$-dimensional manifold M is a **contact manifold** if there exists a 1-form η, called a **contact 1-form**, on M such that

$$\eta \wedge (d\eta)^n \neq 0 \,.$$

For every choice of contact 1-form η there exists a unique vector field K_η, called the **Reeb vector field**, that satisfies

$$\eta(K_\eta) = 1 \quad \text{and} \quad K_\eta \lrcorner \, d\eta = 0 \,.$$

Sasakian geometry (2)

Contact structures(2)

Example:
Consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates

$$(x^1, \ldots, x^n; y^1, \ldots, y^n; z).$$

Contact 1-form, Reeb vector field and Riemannian metric:

$$\eta = dz - \sum_i^n y^i dx^i \,,$$

$$K_\eta = \frac{\partial}{\partial z}\,,$$

$$g = \eta^2 + \sum_i^n \Big((dx^i)^2 + (dy^i)^2 \Big)\ .$$

Sasakian geometry (3)

A simple and direct definition of the Sasakian structures is the following:

A compact Riemannian manifold (Y, g) is **Sasakian** if and only if its metric cone $(X = C(Y) \cong \mathbb{R}_+ \times Y \, , \, \bar{g} = dr^2 + r^2 g)$ is Kähler.

Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line $\mathbb{R}_+$. The Sasakian manifold (Y, g) is naturally isometrically embedded into the metric cone via the inclusion $Y = \{r = 1\} = \{1\} \times Y \subset C(Y)$.

Sasakian geometry (4)

Let us denote by

$$\tilde{K} \equiv \mathcal{J}\Big(r\frac{\partial}{\partial r}\Big),$$

where $\mathcal{J}$ is the complex structure on the cone manifold.
$\tilde{K} - i\mathcal{J}\tilde{K}$ is a holomorphic vector field on $C(Y)$ and the restriction K of $\tilde{K}$ to $Y \subset C(Y)$ is the **Reeb** vector field on Y.
The Reeb vector field K is a Killing vector on (Y, g) , has unit length and, in particular, is nowhere zero.

Let Y be a Sasaki-Einstein manifold of dimension $\dim_{\mathbb{R}} Y = 2n - 1$ and its Kähler cone $X = C(Y)$ is of dimension $\dim_{\mathbb{R}} X = 2n$, ($\dim_{\mathbb{C}} X = n$).

Sasakian geometry (5)

Sasaki-Einstein geometry is naturally “sandwiched” between two Kähler-Einstein geometries as shown in the following proposition:

Let (Y, g) be a Sasaki manifold of dimension $2n - 1$. Then the following are equivalent

(1) (Y, g) is Sasaki-Einstein with $Ric_g = 2(n-1)g$;

(2) The Kähler cone $(C(Y), \bar{g})$ is Ricci-flat, $Ric_{\bar{g}} = 0$;

(3) The transverse Kähler structure to the Reeb foliation $\mathcal{F}_K$ is Kähler-Einstein with $Ric^T = 2ng^T$.

Sasakian geometry (6)

The Kähler form ω is an exact 2-form and homogeneous degree 2 under the Euler vector $r\frac{\partial}{\partial r}$

$$\omega = -\frac{1}{2}d(r^2\eta) = -rdr \wedge \eta - \frac{1}{2}r^2 d\eta \quad ,$$

$$\mathcal{L}_{r\frac{\partial}{\partial r}}\omega = 2\omega,$$

where η is the Sasakian 1-form of Y. It lifts to $C(Y)$ as

$$\eta = \mathcal{J}\left(\frac{dr}{r}\right) = i(\partial - \bar{\partial})\log r \ = \ -2i\partial\bar{\partial}\log r.$$

Sasakian geometry (7)

Note that the Reeb vector $\tilde{K}$ is dual to the 1-form $r^2\eta$.
The Kähler form ω can be written as

$$\omega = \frac{1}{2} i\partial\bar{\partial} r^2,$$

which means that

$$F = \frac{r^2}{4},$$

is the Kähler potential.

Symplectic approach (1)

Let (y, ϕ) be the symplectic coordinates on X.
If (X, ω) is toric, the standard n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ acts effectively on X

$$\tau : \mathbb{T}^n \to Diff(X, \omega),$$

preserving the Kähler form. $\partial/\partial\phi_i$ generate the $\mathbb{T}^n$ action, ϕ_i being the angular coordinates along the orbit of the torus action $\phi_i \sim \phi_i + 2\pi$. $\mathbb{T}^n$-invariant Kähler metric on X is

$$ds^2 = G_{ij} dy_i dy_j + G^{ij} d\phi_i d\phi_j,$$

where G_{ij} is the Hessian of the symplectic potential $G(y)$ in the y coordinates

$$G_{ij} = \frac{\partial^2 G}{\partial y_i \partial y_j} \quad , \quad 1 \leq i, j \leq n,$$

and $G^{ij} = (G_{ij})^{-1}$.

Symplectic approach (2)

The almost complex structure is

$$\mathcal{J} = \begin{pmatrix} 0 & -G^{ij} \\ G_{ij} & 0 \end{pmatrix},$$

and the symplectic (Kähler) form is $\omega = dy_i \wedge d\phi_i$.
Associated to (X, ω, τ) there is a moment map $\mu : X \to \mathbb{R}^n$

$$\mu(y, \phi) = y,$$

i.e. the projection on the action coordinates:

$$y_i = -\frac{1}{2}\left\langle r^2\eta, \frac{\partial}{\partial \phi_i}\right\rangle.$$

Symplectic approach (3)

Let us write the Reeb vector in the form:

$$\tilde{K} = b_i \frac{\partial}{\partial \phi_i}.$$

In the symplectic coordinates (y, ϕ) we have

$$r \frac{\partial}{\partial r} = 2 y_i \frac{\partial}{\partial y_i},$$

and the components of the Reeb vector are $b_i = 2 G_{ij} y_j$.

Symplectic approach (4)

The moment map exhibits the Kähler cone as a Lagrangian fibration over a strictly convex rational polyhedral cone $\mathcal{C} \subset \mathbb{R}^n$ by forgetting the angular coordinates ϕ_i

$$\mathcal{C} \left\{ y \in \mathbb{R}^n | l_a(y) \geqslant 0 \quad , \quad a = 1, \ldots, d \right\},$$

with the linear function $l_a(y) = (y, v_a)$, where v_a are the inward pointing normal vectors to the d facets of the polyhedral cone. The set of vectors $\{v_a\}$

$$v_a = v_a^i \frac{\partial}{\partial \phi_i} \quad , \quad v_a^i \in \mathbb{Z},$$

is called a *toric data*.

Delzant construction (1)

The image of X under the moment map μ is a certain kind of convex rational polytope in $\mathbb{R}^n$ called Delzant polytope.

A convex polytope P in $\mathbb{R}^n$ is **Delzant** [Delzant, 1988] if

(a) there are n edges meeting at each vertex p;

(b) the edges meeting at the vertex p are rational, i.e. each edge is of the form $1 + tu_i$, $0 \leq t \leq \infty$ where $u_i \in \mathbb{Z}^n$;

(c) the $u_i, \ldots, u_n$ in (b) can be chosen to be a basis of $\mathbb{Z}^n$.

Delzant construction associates to every Delzant polytope $P \subset \mathbb{R}^n$ a closed connected symplectic manifold (M, ω) together with the Hamiltonian $\mathbb{T}^n$ action and the moment map μ.

Delzant construction (2)

Using the Delzant construction the general symplectic potential has the following form in terms of the toric data:

$$G = G^{can} + G_b + h,$$

where

$$G^{can} = \frac{1}{2}\sum_a l_a(y)\log l_a(y),$$

$$G_b = \frac{1}{2}\sum_a l_b(y)\log l_b(y) - \frac{1}{2} l_\infty(y)\log l_\infty(y),$$

with $l_b(y) = (b, y)$, $l_\infty(y) = \sum_a (v_a, y)$ and h is a homogeneous degree one function of variables y_i

$$h = \lambda_i y_i + t,$$

λ_i, t being some constants.

Complex approach (1)

The standard complex coordinates are w_i on $\mathbb{C}\backslash\{0\}$. Log complex coordinates are $z_i = \log w_i = x_i + i\phi_i$ and in these complex coordinates the metric is

$$ds^2 = F_{ij}dx_i dx_j + F_{ij}d\phi_i d\phi_j,$$

where F_{ij} is the Hessian of the Kähler potential. Note also that in the complex coordinates z_i the complex structures and the Kähler form are:

$$\mathcal{J} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \quad , \quad \omega = \begin{pmatrix} 0 & F_{ij} \\ -F_{ij} & 0 \end{pmatrix}.$$

Complex approach (2)

The symplectic potential G and Kähler potential F are related by the Legendre transform

$$F(x) = \left(y_i \frac{\partial G}{\partial y_i} - G \right) \ (y = \partial F / \partial x).$$

Therefore F and G are Legendre dual to each other

$$F(x) + G(y) = \sum_j \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial y_j} \text{ at } x_i = \frac{\partial G}{\partial y_i} \text{ or } y_i = \frac{\partial F}{\partial x_i}.$$

It follows from that $F_{ij} = G^{ij}$ $(y = \partial F / \partial x)$.

Complex approach (3)

The $(n, 0)$ holomorphic form of the Ricci-flat metric on the Calabi-Yau cone is

$$dV = e^{i\alpha}(\det F_{ij})^{1/2} dz_1 \wedge \cdots \wedge dz_n,$$

with α a phase space which is fixed by requiring that dV is a closed form. The complex coordinates can be chosen such that

$$dV = e^{x_1 + i\phi_1} dz_1 \wedge \cdots \wedge dz_n = dw_1 \wedge \cdots \wedge dw_n / (w_2 \ldots w_n).$$

The Kähler potential F is obtain by the Legendre transform

$$F(x) = \frac{r^2}{4} = \frac{1}{2} \sum_i b_i y_i - t,$$

Detailed analysis shows that the constant t must be set to zero..

Hidden symmetries on Sasaki-Einstein spaces (1)

The Killing forms of the toric Sasaki-Einstein manifold Y are described by the special Killing forms

$$\Theta_k = \eta \wedge (d\eta)^k \quad , \quad k = 0, 1, \cdots, n-1.$$

Besides these Killing forms, there are $n-1$ closed conformal Killing forms (also called $*$-Killing forms)

$$\Phi_k = (d\eta)^k \quad , \quad k = 1, \cdots, n-1.$$

Hidden symmetries on Sasaki-Einstein spaces (2)

Moreover in the case of the Calabi-Yau cone, the holonomy is $SU(n)$ and there are *two additional* Killing forms of degree n. In order to write explicitly these additional Killing forms we shall express the volume form of the metric cone in terms of the Kähler form

$$d\mathcal{V} = \frac{1}{n!}\omega^n .$$

Here ω^n is the wedge product of ω with itself n times. The volume of a Kähler manifold can be also written as

$$d\mathcal{V} = \frac{i^n}{2^n}(-1)^{n(n-1)/2} dV \wedge \overline{dV} ,$$

where dV is the complex volume holomorphic $(n, 0)$ form of $C(Y)$. The additional (real) Killing forms are given by the real respectively the imaginary part of the complex volume form. [Semmelmann, 2003]

Hidden symmetries on Sasaki-Einstein spaces (3)

In order to extract the corresponding additional Killing forms of the Einstein-Sasaki spaces we make use of the fact that for any p-form ψ on the space Y we can define an associated $p+1$-form ψ^C on the cone $C(Y)$:

$$\psi^C := r^p dr \wedge \psi + \frac{r^{p+1}}{p+1} d\psi.$$

ψ^C is parallel if and only if ψ is a special Killing form with constant $c = -(p+1)$.

Y(p,q) spaces (1)

Infinite family $Y(p,q)$ of Einstein-Sasaki metrics on $S^2 \times S^3$ provides supersymmetric backgrounds relevant to the AdS/CFT correspondence. The total space $Y(p,q)$ of an S^1-fibration over $S^2 \times S^2$ with relative prime winding numbers p and q is topologically $S^2 \times S^3$.

Explicit local metric of the 5-dimensional $Y(p,q)$ manifold given by the line element [Gauntlett, Martelli, Sparks, Waldram, 2004]

$$
\begin{aligned}
ds^2_{ES} = {} & \frac{1 - c\,y}{6}(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{w(y)q(y)}dy^2 \\
& + \frac{q(y)}{9}(d\psi - \cos\theta\, d\phi)^2 \\
& + w(y)\left[d\alpha + \frac{ac - 2y + c\,y^2}{6(a - y^2)}[d\psi - \cos\theta\, d\phi]\right]^2 ,
\end{aligned}
$$

where

Y(p,q) spaces (2)

$$w(y) = \frac{2(a - y^2)}{1 - cy} \quad , \quad q(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2}$$

and a, c are constants. The constant c can be rescaled by a diffeomorphism and in what follows we assume $c = 1$.
The coordinate change $\alpha = -\frac{1}{6}\beta - \frac{1}{6}c\,\psi'$, $\psi = \psi'$ takes the line element to the following form (with $p(y) = w(y)\,q(y)$)

$$\begin{aligned} ds^2_{ES} = {} & \frac{1-y}{6}(d\theta^2 + \sin^2\theta\,d\phi^2) + \frac{1}{p(y)}dy^2 \\ & + \frac{p(y)}{36}(d\beta + \cos\theta\,d\phi)^2 \\ & + \frac{1}{9}[d\psi' - \cos\theta\,d\phi + y(d\beta + \cos\theta\,d\phi)]^2 \,, \end{aligned}$$

Y(p,q) spaces (3)

For

$$0 < \alpha < 1\,,$$

we can take the range of the angular coordinates (θ, Φ, Ψ) to be $0 \le \theta \le 2\pi\,, 0 \le \Phi \le 2\pi\,, 0 \le \Psi \le 2\pi$. Choosing $0 < a < 1$ the roots y_i of the cubic equation

$$a - 3y^2 + 2y^3 = 0\,,$$

are real, one negative (y_1) and two positive (y_2, y_3). If the smallest of the positive roots is y_2, one can take the range of the coordinate y to be

$$y_1 \le y \le y_2\,.$$

Y(p,q) spaces (4)

For this particular space we take the complex coordinates

$$z^1 = 3\ln r + \ln\sin\theta + \frac{1}{2}\ln\frac{p(y)(1-y)}{2} + \mathrm{i}\psi'$$

$$z^2 = \frac{1}{3\sqrt{3}}\left(\ln\tan\frac{\theta}{2} + \mathrm{i}\phi\right)$$

$$z^3 = \frac{1}{6}\ln\left(\frac{1}{\sin\theta}\sqrt{(y-y_1)^{-\frac{1}{y^1}}(y_2-y)^{-\frac{1}{y^2}}(y_3-y)^{-\frac{1}{y^3}}}\right) - \mathrm{i}\alpha - \frac{1}{6}\mathrm{i}\psi'$$

Y(p,q) spaces (5)

One can write

$$ds^2_{ES} = ds^2_{EK} + (\frac{1}{3}d\psi' + \sigma)^2$$

The Sasakian 1-form of the $Y(p,q)$ space is

$$\eta = \frac{1}{3}d\psi' + \sigma\,,$$

with

$$\sigma = \frac{1}{3}[-\cos\theta\, d\phi + y(d\beta + \cos\theta\, d\phi)]\,.$$

connected with local Kähler form Ω_{EK}.
This form of the metric with the 1-form η is the standard one for a locally Einstein-Sasaki metric with $\frac{\partial}{\partial\psi'}$ the Reeb vector field.

Y(p,q) spaces (6)

The local Kähler and holomorphic $(2,0)$ form for ds^2_{EK} are

$$\Omega_{EK} = \frac{1-y}{6}\sin\theta d\theta \wedge d\phi + \frac{1}{6}dy \wedge (d\beta + \cos\theta d\phi)$$

$$dV_{EK} = \sqrt{\frac{1-y}{6p(y)}}(d\theta + i\sin\theta d\phi) \wedge \left[dy + i\frac{p(y)}{6}(d\beta + \cos\theta d\phi)\right]$$

Y(p,q) spaces (7)

From the isometries $SU(2) \times U(1) \times U(1)$ the momenta P_ϕ, P_ψ, P_α and the Hamiltonian describing the geodesic motions are conserved. P_ϕ is the third component of the $SU(2)$ angular momentum, while P_ψ and P_α are associated with the $U(1)$ factors. Additionally, the total $SU(2)$ angular momentum given by

$$J^2 = P_\theta^2 + \frac{1}{\sin^2\theta}(P_\phi + \cos\theta P_\psi)^2 + P_\Psi^2 \,,$$

is also conserved.

Y(p,q) spaces (8)

Specific conserved quantities for Einstein-Sasaki spaces (1)

First of all from the 1-form η

$$\begin{aligned}\Psi &= \eta \wedge d\eta \\ &= \frac{1}{9}[(1-y)\sin\theta\, d\theta \wedge d\phi \wedge d\psi' + dy \wedge d\beta \wedge d\psi' \\ &\quad + \cos\theta\, dy \wedge d\phi \wedge d\psi' - \cos\theta\, dy \wedge d\beta \wedge d\phi \\ &\quad + (1-y)y \sin\theta\, d\beta \wedge d\theta \wedge d\phi]\,.\end{aligned}$$

is a special Killing form. Let us note also that

$$\Psi_k = (d\eta)^k\,, \quad k = 1, 2\,,$$

are closed conformal Killing forms ($\star$-Killing forms).

Y(p,q) spaces (9)

Specific conserved quantities for Einstein-Sasaki spaces (2)

On the Calabi-Yau manifold the Kähler form is

$$\Omega_{cone} = r dr \wedge \eta + r^2 \Omega_{EK} \,.$$

and the holomorphic $(3,0)$ form is

$$\begin{aligned} dV_{cone} &= e^{\psi'} r^2 dV_{EK} \wedge [dr + ir \wedge \eta] \\ &= e^{\psi'} r^2 \sqrt{\frac{1-y}{6p(y)}} (d\theta + i \sin\theta d\phi) \\ &\quad \wedge \left[dy + i \frac{p(y)}{6} (d\beta + \cos\theta d\phi) \right] \\ &\quad \wedge \left[dr + i \frac{r}{3} [y d\beta + d\psi' - (1-y) \cos\theta d\phi \right] \end{aligned}$$

Y(p,q) spaces (10)

Specific conserved quantities for Einstein-Sasaki spaces (3)

The additional Killing 3-forms of the $Y(p,q)$ spaces are extracted from the volume form dV_{cone}.

Using the the 1-1 correspondence between special Killing p-forms on M_{2n+1} and parallel $(p+1)$-forms on the metric cone $C(M_{2n+1})$ for $p=2$ we get the following additional Killing 2-forms of the $Y(p,q)$ spaces written as real forms:

$$\Xi = \mathrm{Re}\,\omega^M = \sqrt{\frac{1-y}{6\,p(y)}} \times \Big(\cos\psi'\Big[-dy\wedge d\theta + \frac{p(y)}{6}\sin\theta\, d\beta\wedge d\phi\Big] - \sin\psi'\Big[-\sin\theta\, dy\wedge d\phi - \frac{p(y)}{6}d\beta\wedge d\theta + \frac{p(y)}{6}\cos\theta\, d\theta\wedge d\phi\Big]\Big)$$

Y(p,q) spaces (11)

Specific conserved quantities for Einstein-Sasaki spaces (4)

$$\Upsilon = \mathrm{Im}\,\omega^M = \sqrt{\frac{1-y}{6\,p(y)}}$$
$$\times\left(\sin\psi'\left[-dy\wedge d\theta + \frac{p(y)}{6}\sin\theta\, d\beta\wedge d\phi\right]\right.$$
$$+\cos\psi'\left[-\sin\theta\, dy\wedge d\phi - \frac{p(y)}{6}d\beta\wedge d\theta\right.$$
$$\left.\left.+\frac{p(y)}{6}\cos\theta\, d\theta\wedge d\phi\right]\right)$$

Y(p,q) spaces (12)

Specific conserved quantities for Einstein-Sasaki spaces (5)

The Stäckel-Killing tensors associated with the Killing forms $\Psi\,,\Xi\,,\Upsilon$ are constructed as usual. Together with the Killing vectors P_ϕ, P_ψ, P_α and the total angular momentum J^2 these Stäckel-Killing tensors provide the superintegrability of the $Y(p,q)$ geometries.

Outlook

- Complete integrability of geodesic equations
- Separability of Hamilton-Jacobi, Klein-Gordon, Dirac equations
- Hidden symmetries of other spacetime structures